Symmetry Breaking and Clock Model Interpolation in 2D Classical O(2) Spin Systems

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Ph.D. Defense

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Outline

Outline

Quantum Chromodynamics (QCD)

- The theory of the strong interaction between color-charged particles
- It is a non-Abelian gauge theory with symmetry group SU(3)
- The action is

$$S = \int d^4x \left[\sum_f \overline{\psi}_i^f \left(i\gamma_\mu D_{ij}^\mu - m_f \delta_{ij} \right) \psi_j^f - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \right]$$

- Vacuum expectation value of observable O can be written as path integral

$$\langle O \rangle = \frac{\int \mathcal{D}A \, \mathcal{D}\overline{\psi} \, \mathcal{D}\psi \; O[A, \overline{\psi}, \psi] \; e^{iS[A,\overline{\psi},\psi]}}{\int \mathcal{D}A \, \mathcal{D}\overline{\psi} \, \mathcal{D}\psi \; e^{iS[A,\overline{\psi},\psi]}}$$

- How to deal with this?

Lattice QCD

- Regularize the theory by discretizing 4D spacetime
- Define quark fields on the lattice sites and gauge fields on the links
- Wick rotate to get Euclidean action and interpret the path integral as a classical partition function
- Equilibrium expectation values can be estimated by Monte Carlo simulation

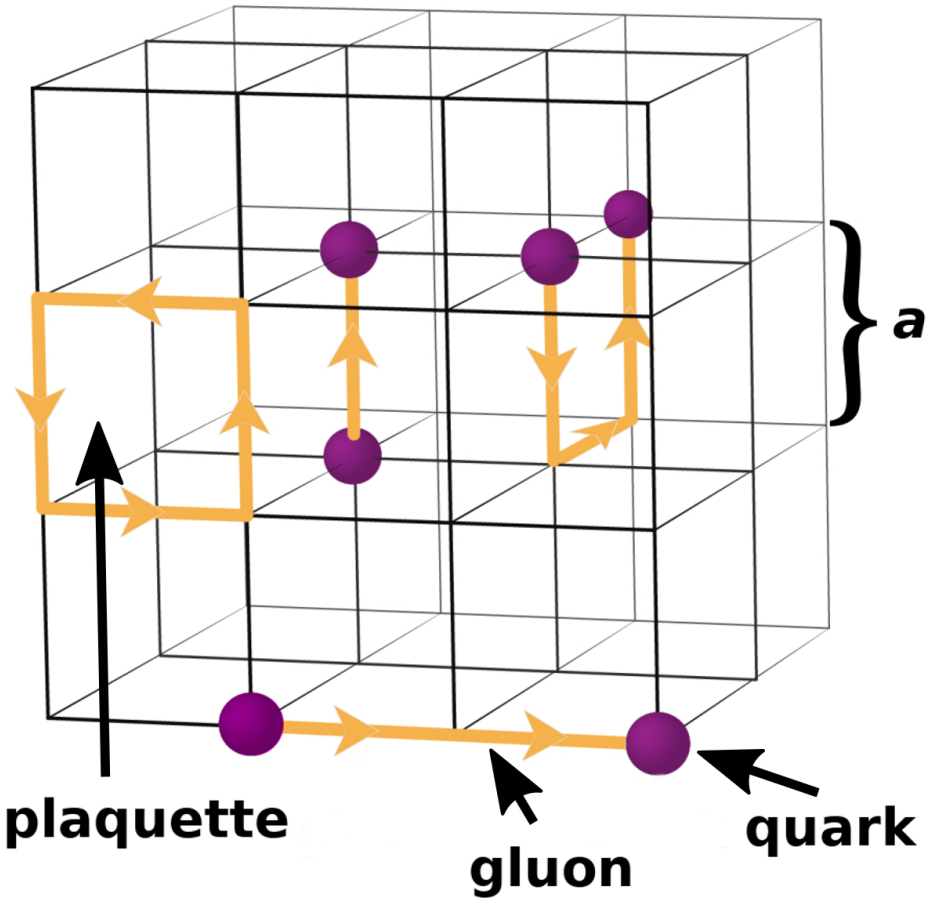


Figure: Image originally from JICFus

Challenges in Lattice QCD

- Distribution weight for gauge fields is proportional to huge (fermion) determinants
- At non-zero baryon density, there is a **sign problem**
- For real-time dynamics, there is a **sign problem**
- We need new approaches

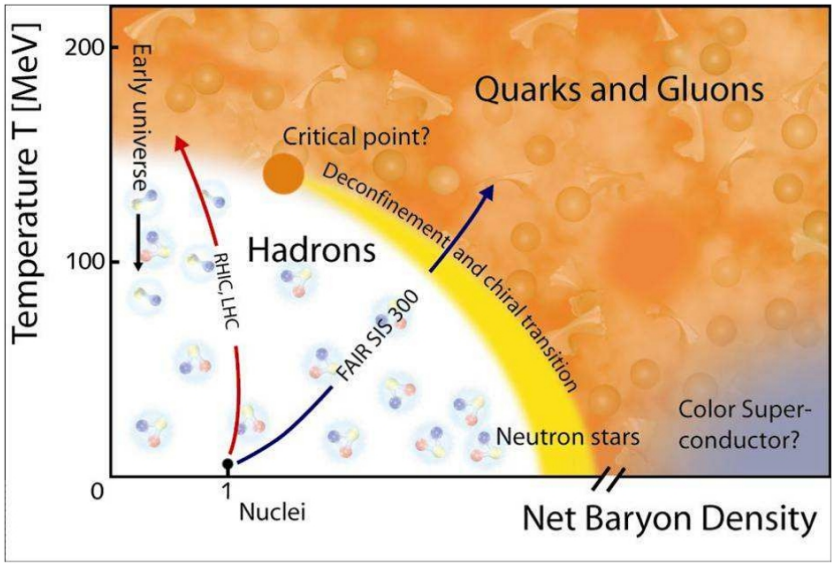
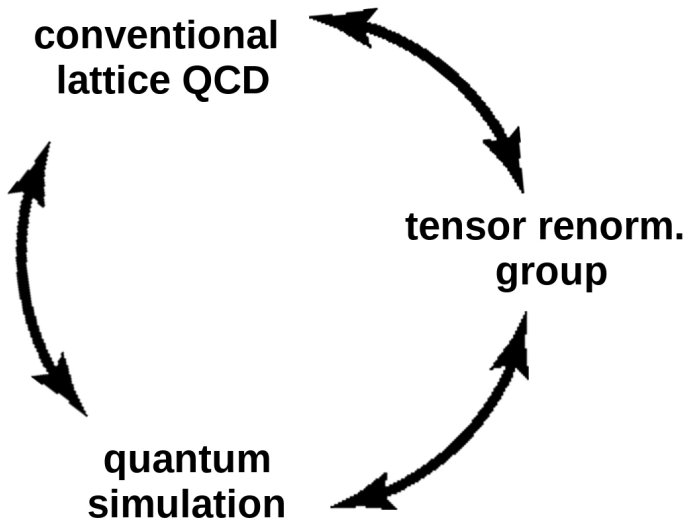


Figure: The conjectured QCD phase diagram. Image: arXiv:1412.0847

A Synergistic Approach

- Quantum simulation:
  - Simulate QFTs at finite density and real time with no sign problem
  - Start with toy models—spin models, then U(1), SU(2)-Higgs, Schwinger model, SU(2) with fermions, ...
- Tensor renormalization group (TRG):
  - Alternative to MCMC approach
  - Use as stepping stone to quantum simulations
- Conventional lattice QCD:
  - Validation and benchmarking
  - Need a lattice codebase that handles arbitrary dimensions and gauge groups



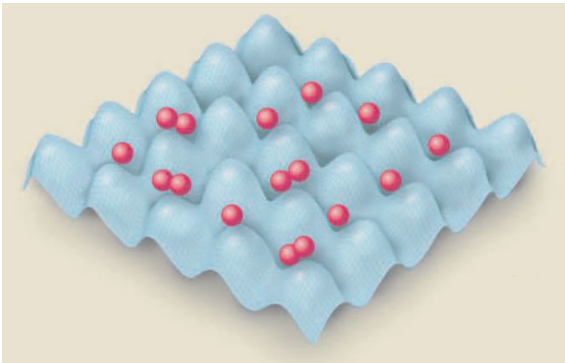
Outline

Quantum Simulation

Digital: Hamiltonian is mapped to a simpler quantum system which is "time-evolved" stroboscopically. *Example:* A universal quantum computer running an algorithm that simulates a discretized QFT

IBM Q, 50-qubit quantum computer

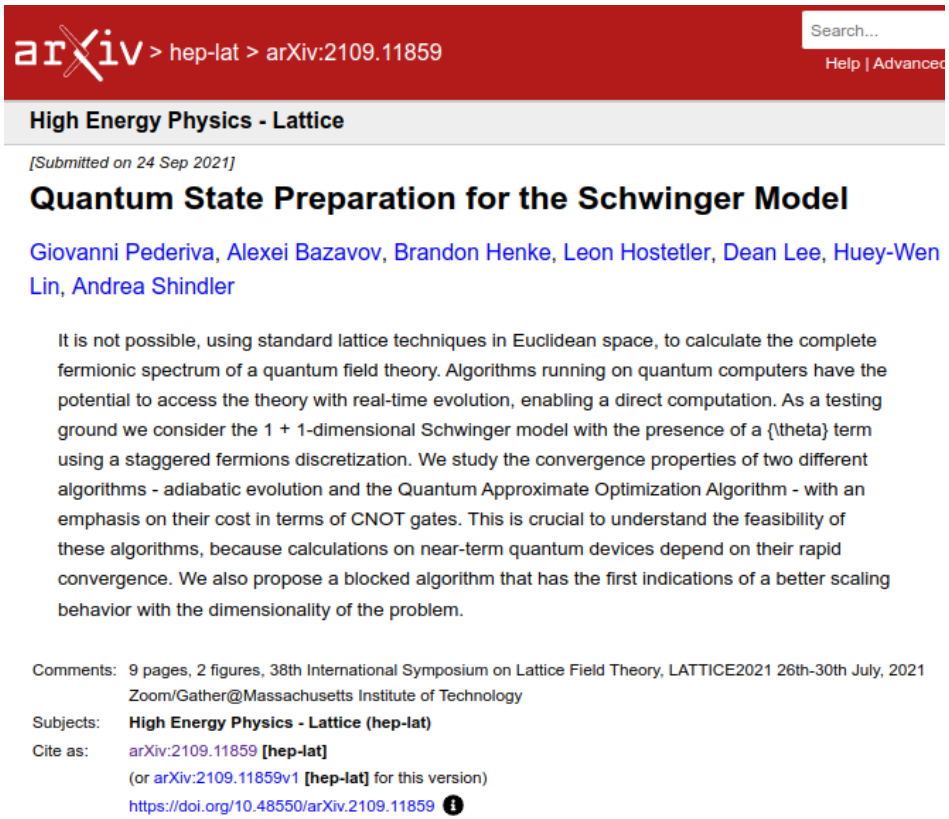
Analog: Hamiltonian is mapped to a simpler quantum system which is allowed to evolve continuously in real-time. *Example:* Atoms hopping around on an optical lattice



Georgescu et. al., Rev. Mod. Phys. 86, 154 (2014)

Digital Quantum Simulation of the Schwinger Model

- Project lead by Giovanni Pederiva
- Schwinger model (QED in 1+1 D) as a toy model for QCD
- We studied state preparation methods: ASP, QAOA, RODEO
- Results are promising for the long-term



arXiv > hep-lat > arXiv:2109.11859

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High Energy Physics - Lattice

[Submitted on 24 Sep 2021]

Quantum State Preparation for the Schwinger Model

Giovanni Pederiva, Alexei Bazavov, Brandon Henke, Leon Hostetler, Dean Lee, Huey-Wen Lin, Andrea Shindler

It is not possible, using standard lattice techniques in Euclidean space, to calculate the complete fermionic spectrum of a quantum field theory. Algorithms running on quantum computers have the potential to access the theory with real-time evolution, enabling a direct computation. As a testing ground we consider the 1 + 1-dimensional Schwinger model with the presence of a {\theta} term using a staggered fermions discretization. We study the convergence properties of two different algorithms - adiabatic evolution and the Quantum Approximate Optimization Algorithm - with an emphasis on their cost in terms of CNOT gates. This is crucial to understand the feasibility of these algorithms, because calculations on near-term quantum devices depend on their rapid convergence. We also propose a blocked algorithm that has the first indications of a better scaling behavior with the dimensionality of the problem.

Comments: 9 pages, 2 figures, 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021 Zoom/Gather@Massachusetts Institute of Technology

Subjects: **High Energy Physics - Lattice (hep-lat)**

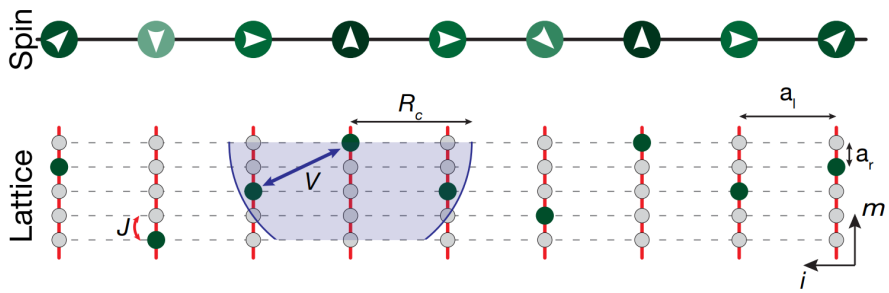
Cite as: arXiv:2109.11859 **[hep-lat]**

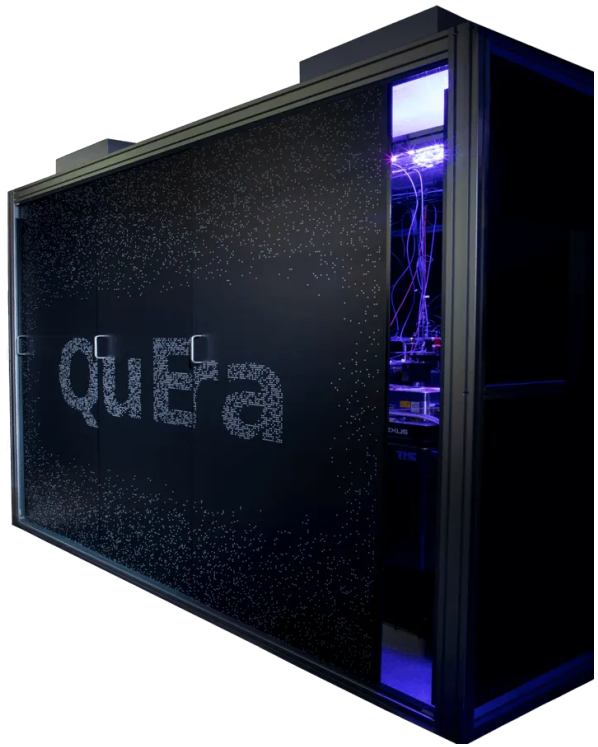
(or arXiv:2109.11859v1 **[hep-lat]** for this version)

https://doi.org/10.48550/arXiv.2109.11859

Analog Quantum Simulation of the Abelian-Higgs Model

- Abelian-Higgs model in 1+1 D is Schwinger model with electron replaced by complex scalar field
- Abelian-Higgs model can be mapped to Rydberg ladder
  - A. Bazavov et. al., Phys. Rev. D 92, 076003 (2015)
  - J. Zhang et. al., Phys. Rev. Lett. 121, 223201 (2018)
  - Y. Meurice, Phys. Rev. D 100, 014506 (2019)
  - Y. Meurice, Phys. Rev. D 104, 094513 (2021)
- Reduces to the classical O(2) model in the limit $\lambda = \infty$ and $g^2 = 0$





Outline

Why Study Classical O(2)-like Spin Models?

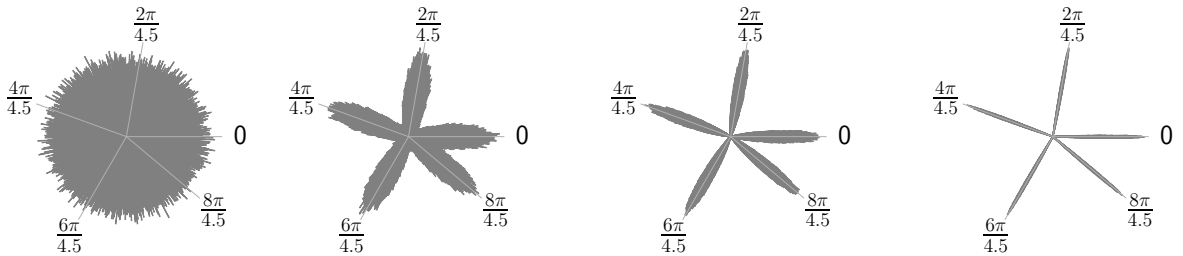
1. Non-trivial limit of the Abelian-Higgs model (scalar QED) in 1+1 D
2. Implementation on an analog simulator may be a first step toward the simulation of more complicated models
3. Can add a symmetry-breaking term to break the O(2) symmetry down to $\mathbb{Z}_q$
   - Study the role of symmetry in spin systems
   - Study $\mathbb{Z}_q$ approximations of continuous $U(1)/O(2)$ symmetry
   - Relevant for "field digitization" of gauge theories
4. Develop tensor renormalization group (TRG) methods in a model that can be validated by conventional MCMC
5. A playground for exploring second-order and BKT phase transitions
6. Test our new codebase

The Extended-O(2) Model

- We consider an extended-O(2) model[1] in 2D with energy function

$$H = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$

- When $h_q = 0$, this is the classic XY model, with a BKT transition
- When $h_q > 0$, the continuous angle φ is forced into the discrete values $\varphi_0 \leq \varphi_{x,k} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$



(a) $h_q = 0$ (b) $h_q = 1$ (c) $h_q = 4$ (d) $h_q = 64$

- When $h_q \to \infty$
  - For $q \in \mathbb{Z}$, this is the ordinary q-state clock model with $\mathbb{Z}_q$ symmetry
  - For $q \notin \mathbb{Z}$, this defines an interpolation of the clock model for noninteger q

[1] JKKN Phys. Rev. B 16, 1217 (1977)

The $h_q \to \infty$ limit[2]

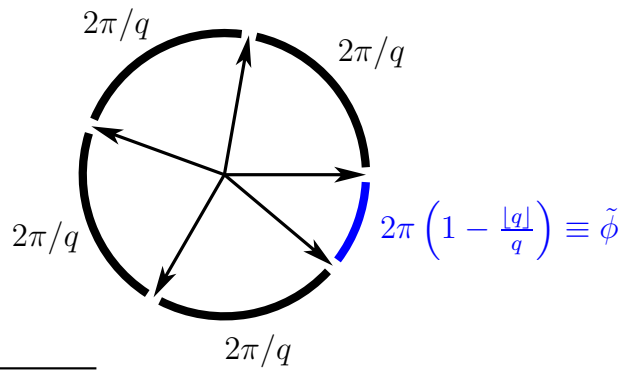
- In the limit $h_q \to \infty$, we can replace the energy function with

$$H_{\text{ext-}q} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)$$

- We directly restrict the previously continuous angles to the discrete values

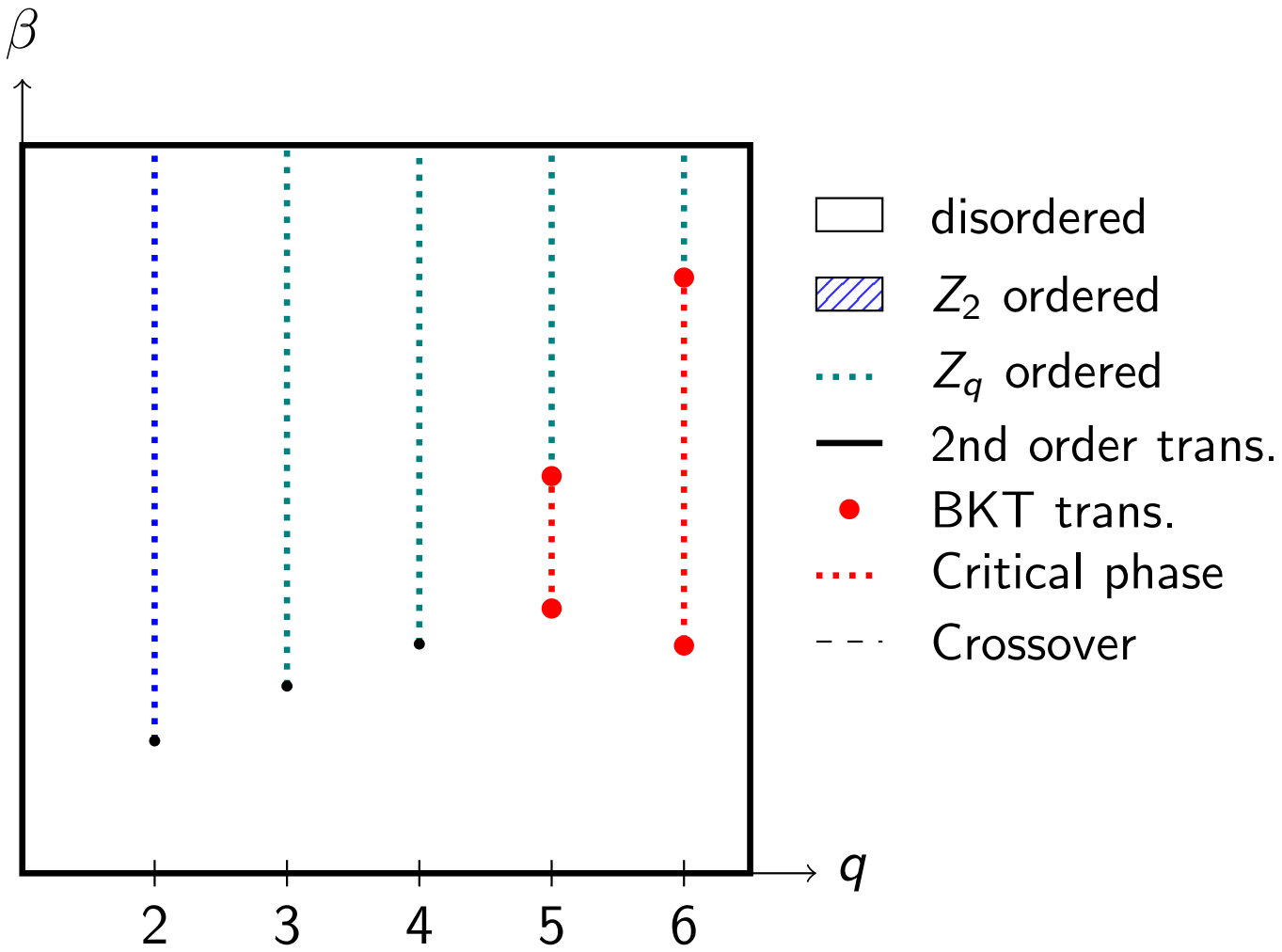
$$\varphi_0 \leq \varphi_{x,k} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$$

- For $q \notin \mathbb{Z}$, divergence from ordinary clock model behavior is driven by the introduction of a "small angle":



[2] Hostetler et. al. PRD 104 (5), 054505 and PoS(LATTICE2021)353

The $h_q \to \infty$ limit[3]



[3]Hostetler et. al. PRD 104 (5), 054505 and PoS(LATTICE2021)353

TRG results at large volume[4]

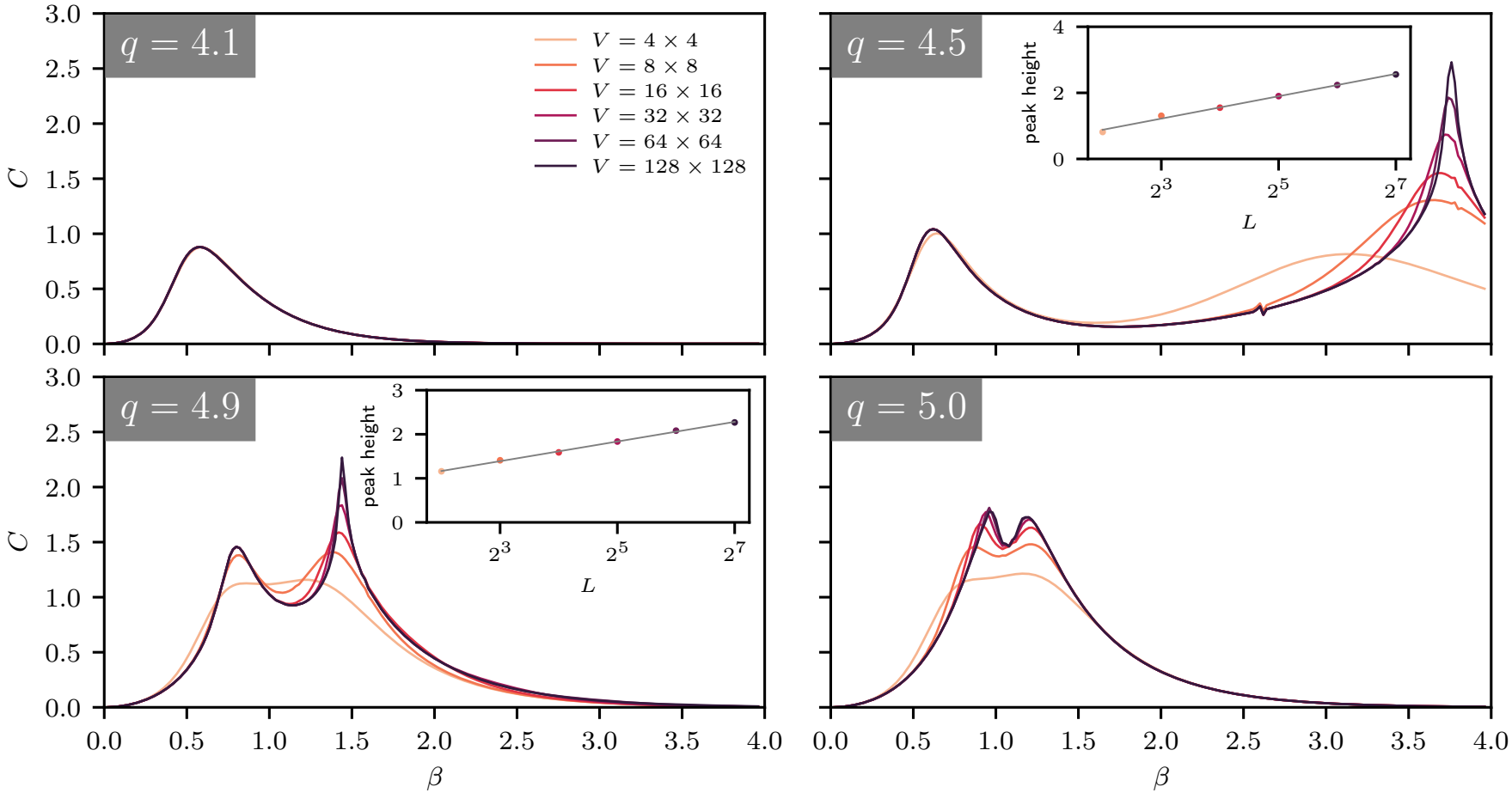
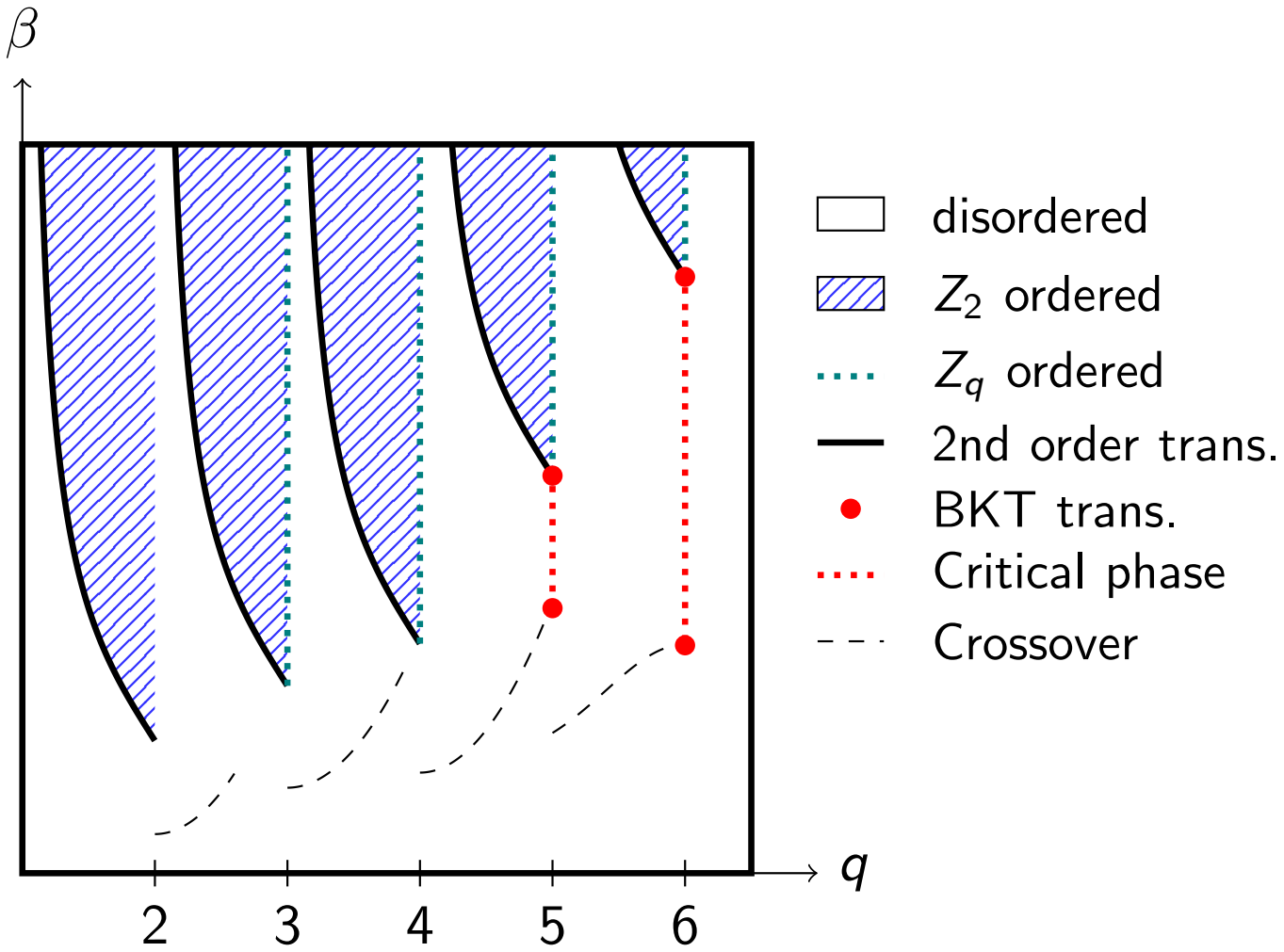


Figure: Specific heat results for the extended-q clock model from TRG for $q = 4.1$, 4.5, 4.9, and 5.0 at volumes from $2^2 \times 2^2$ up to $2^7 \times 2^7$.

[4] Hostetler et. al. PRD 104 (5), 054505 and PoS(LATTICE2021)353

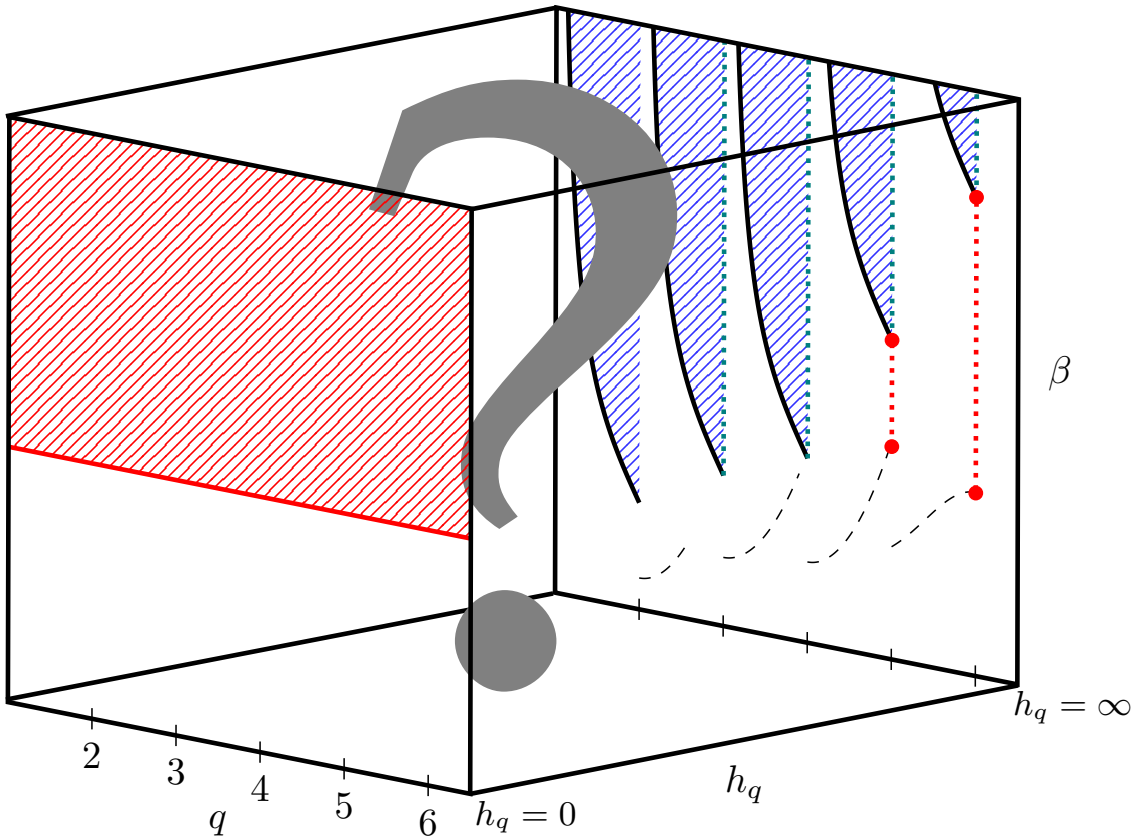
The $h_q \to \infty$ limit[5]



[5] Hostetler et. al. PRD 104 (5), 054505 and PoS(LATTICE2021)353

Phase Diagram

$$H = -\sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - h_q \sum_x \cos(q\varphi_x)$$



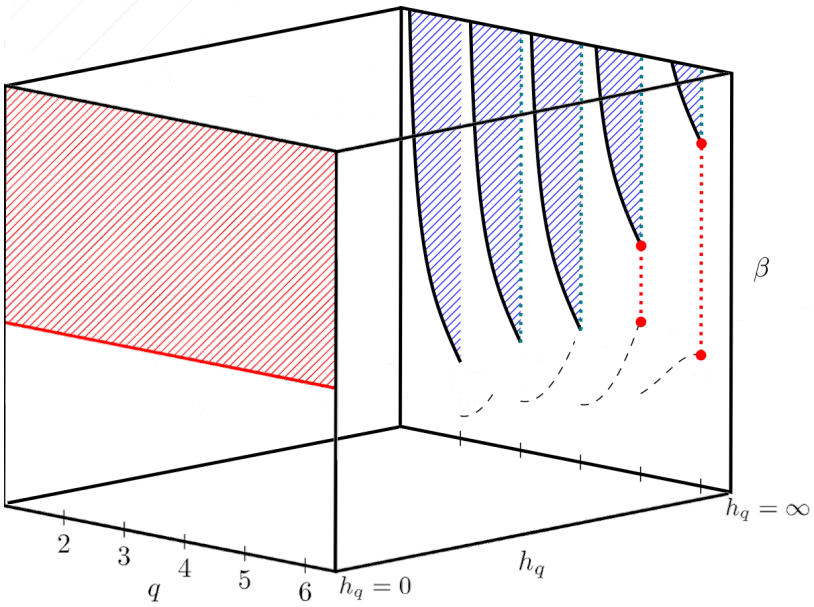
Algorithm Developments

- In the $h_q \to \infty$ limit, the DOF could be treated as discrete
  - Which means we could use an MCMC *heatbath* algorithm
  - We could use a TRG method for large volumes
- The model is more difficult to study at finite h_q
- For finite h_q, the DOF are continuous
  - MCMC heatbath is not an option, so we're left with the Metropolis, which suffers from low acceptance rates and leads to large autocorrelations in this model
  - Furthermore, our TRG method was only designed for the $h_q \to \infty$ limit
- We needed to make some algorithmic developments
  - We implemented a *biased Metropolis heatbath algorithm*[6] (BMHA) which is designed to approach heatbath acceptance rates
  - To explore large volumes, my collaborators implemented a Gaussian quadrature method

[6] A. Bazavov and B. A. Berg, PRD 71, 114506 (2005)

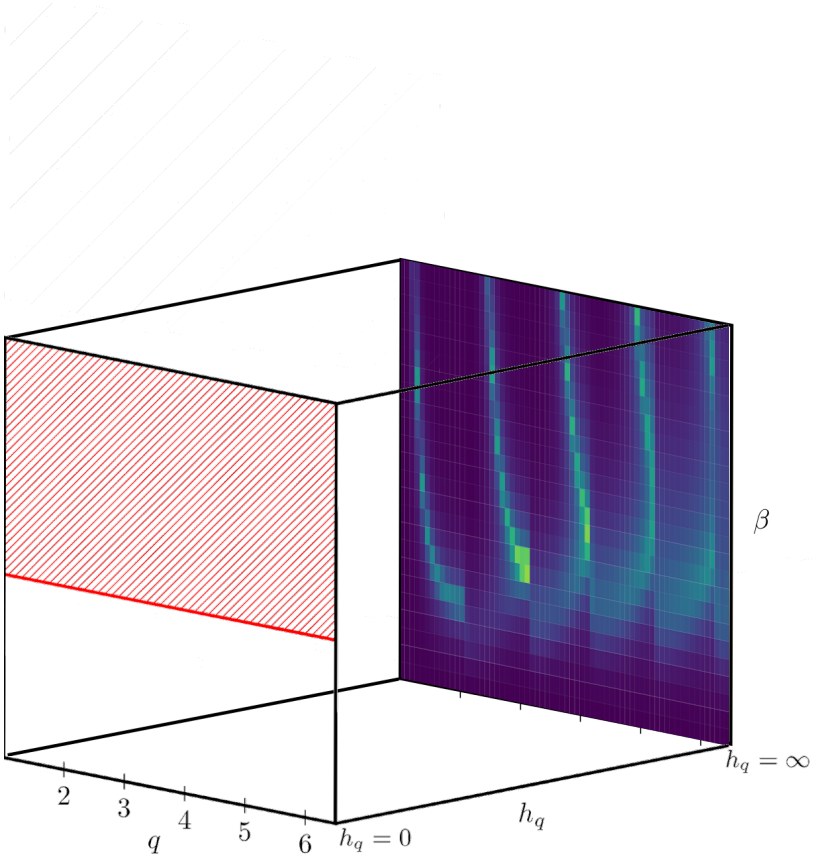
Phase Diagram at Finite-h_q

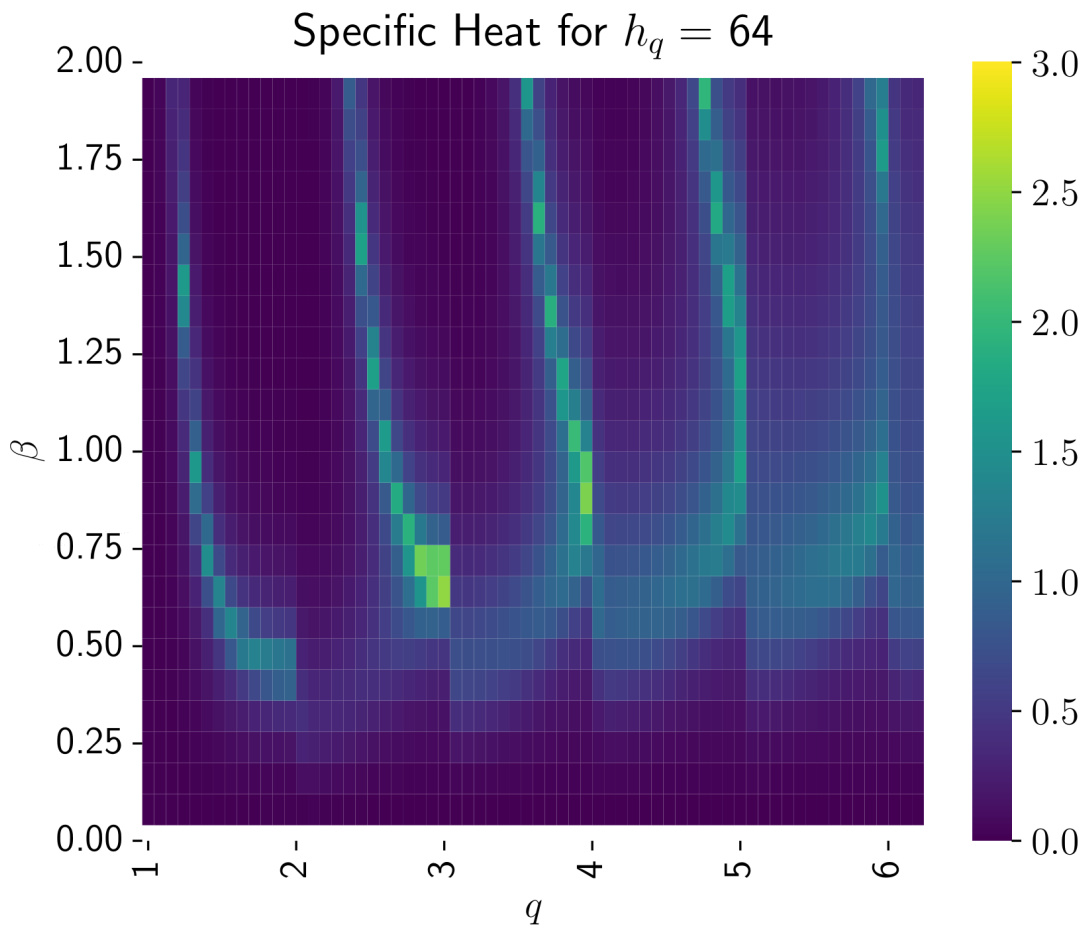
$$S_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$



Specific Heat from TRG with $L = 1024$ and $h_q = 64$

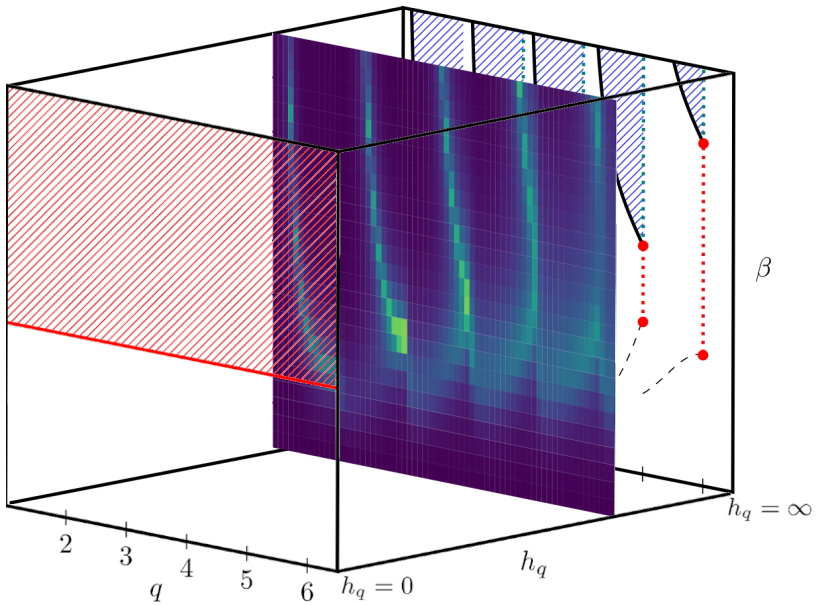
$$H_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$

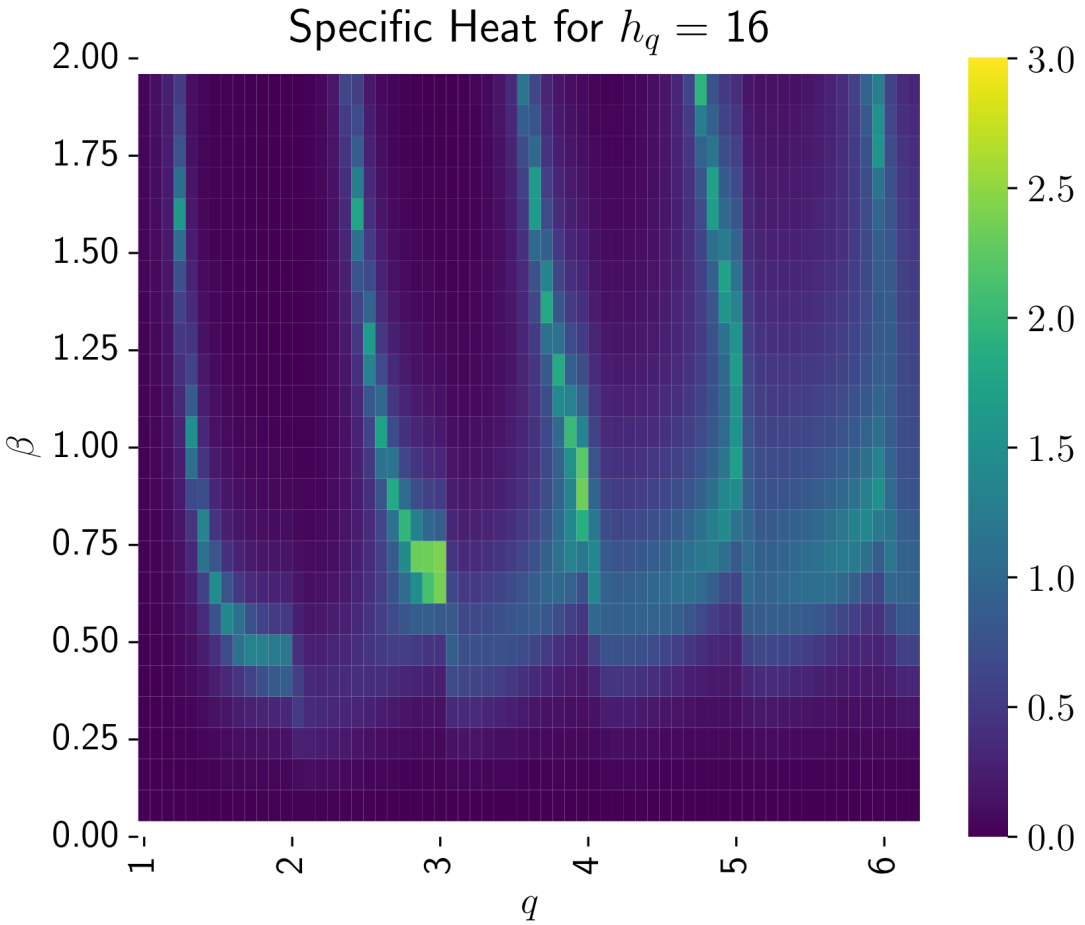




Specific Heat from TRG with $L = 1024$ and $h_q = 16$

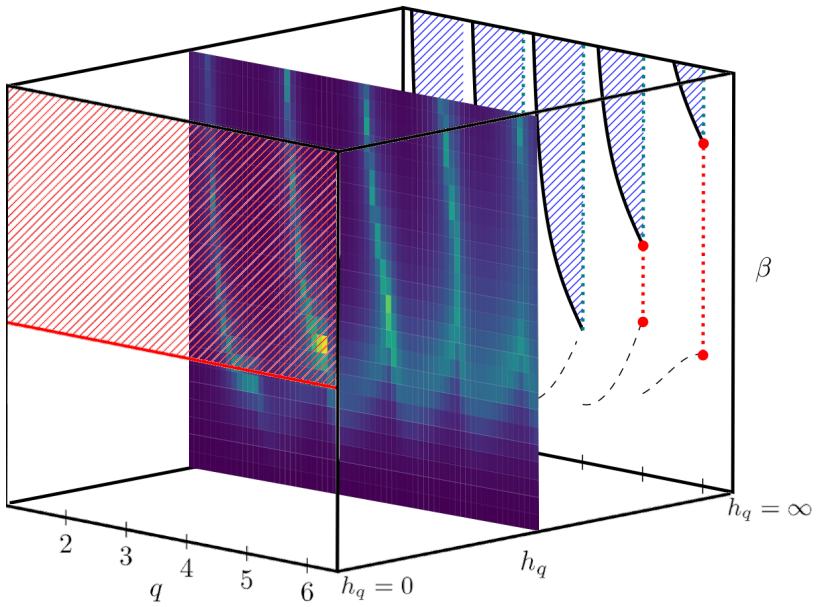
$$H_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$

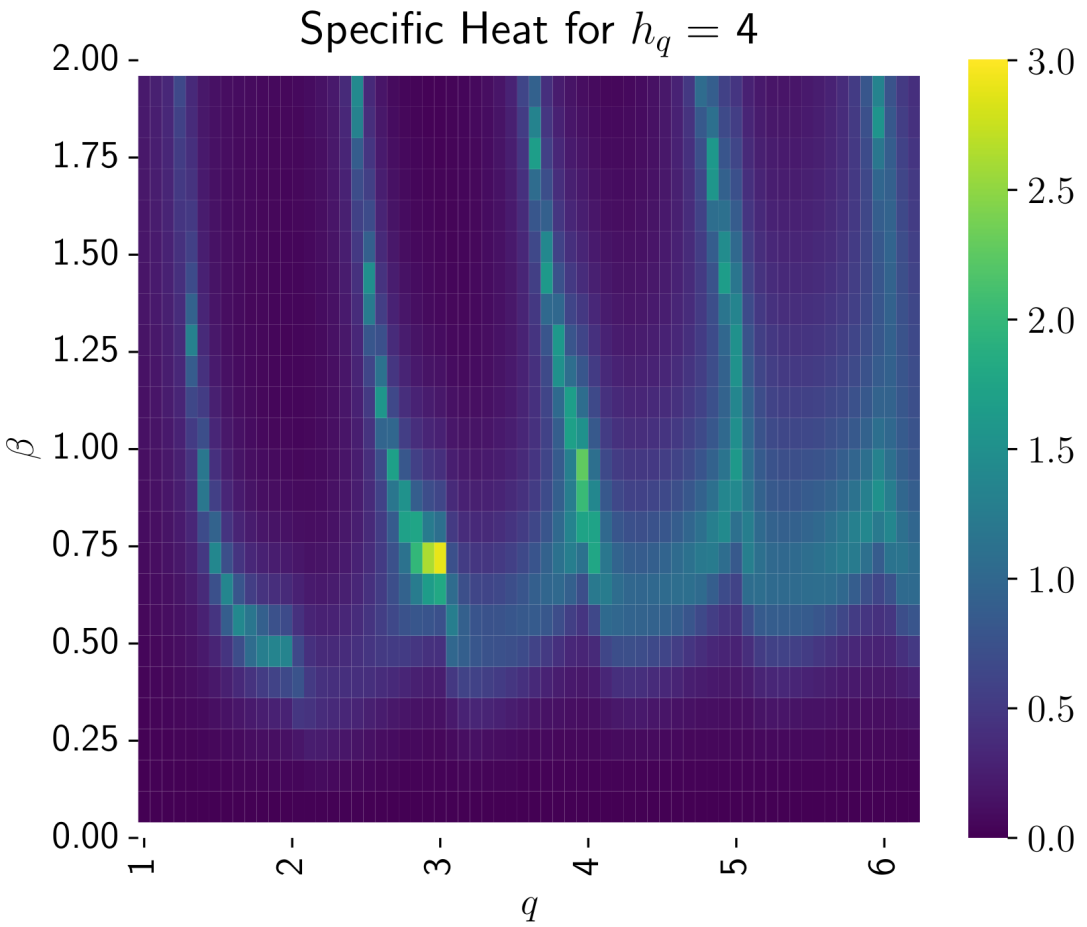




Specific Heat from TRG with $L = 1024$ and $h_q = 4$

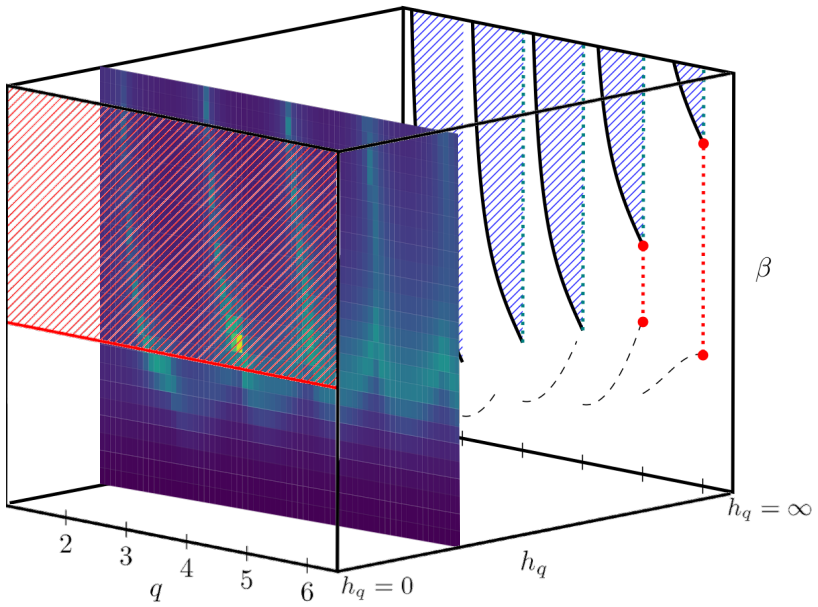
$$H_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$

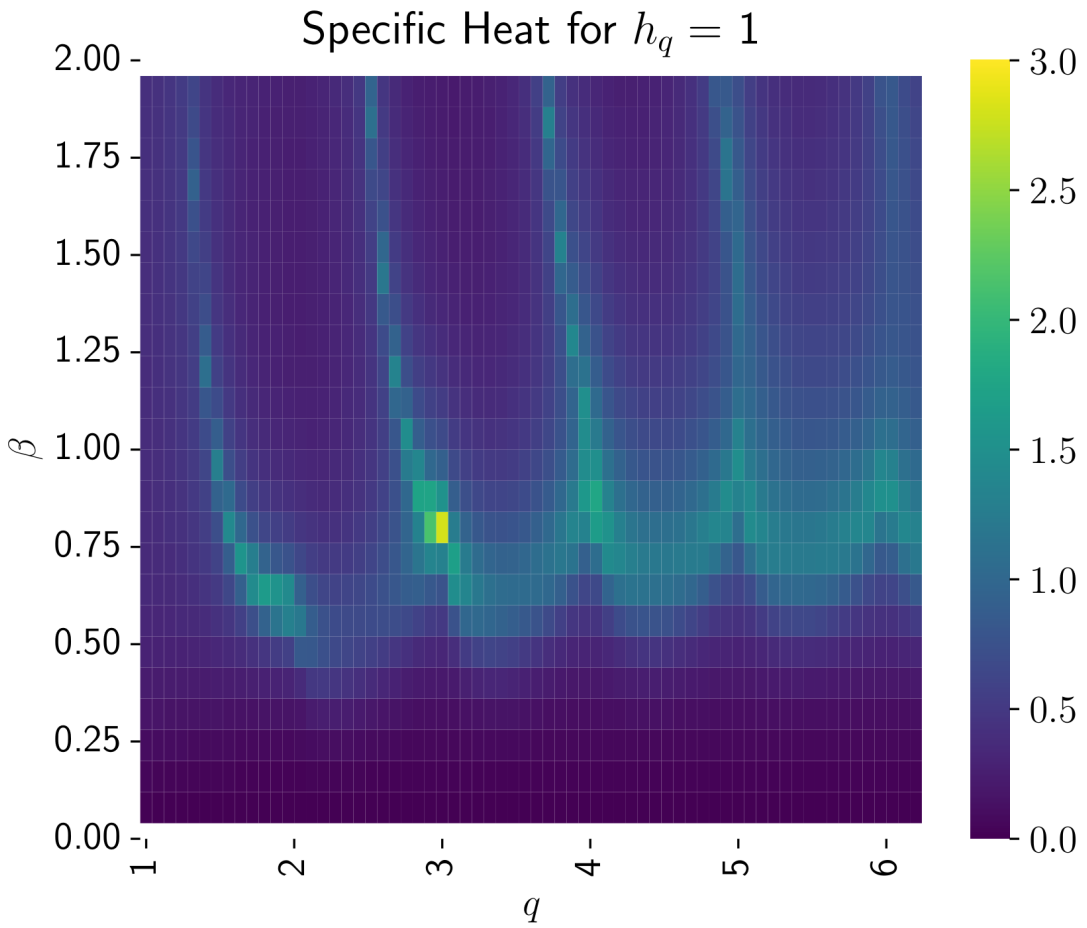




Specific Heat from TRG with $L = 1024$ and $h_q = 1$

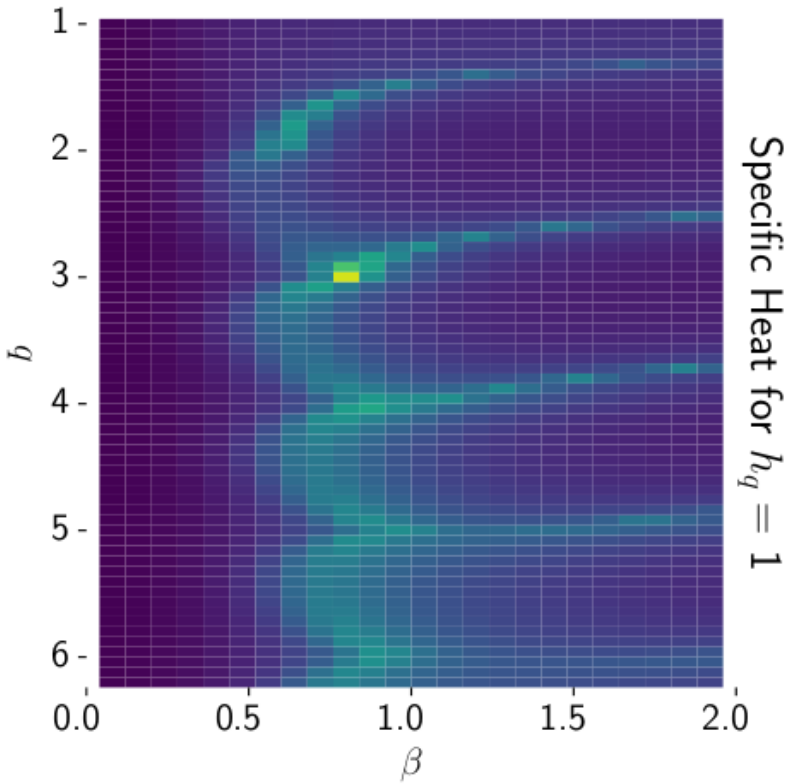
$$H_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$

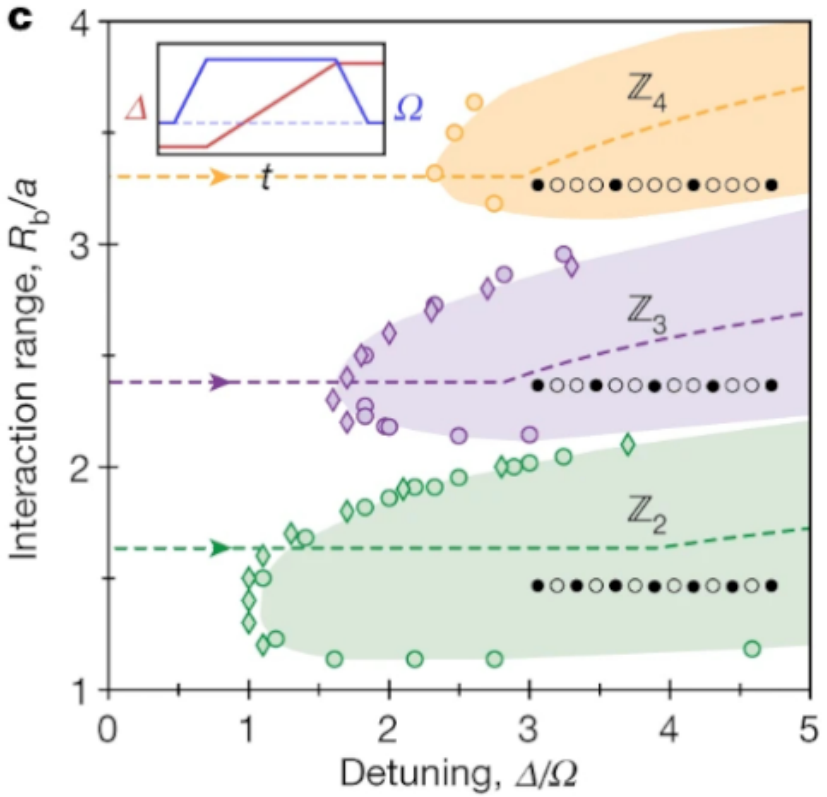




Specific Heat from TRG with $L = 1024$ and $h_q = 1$

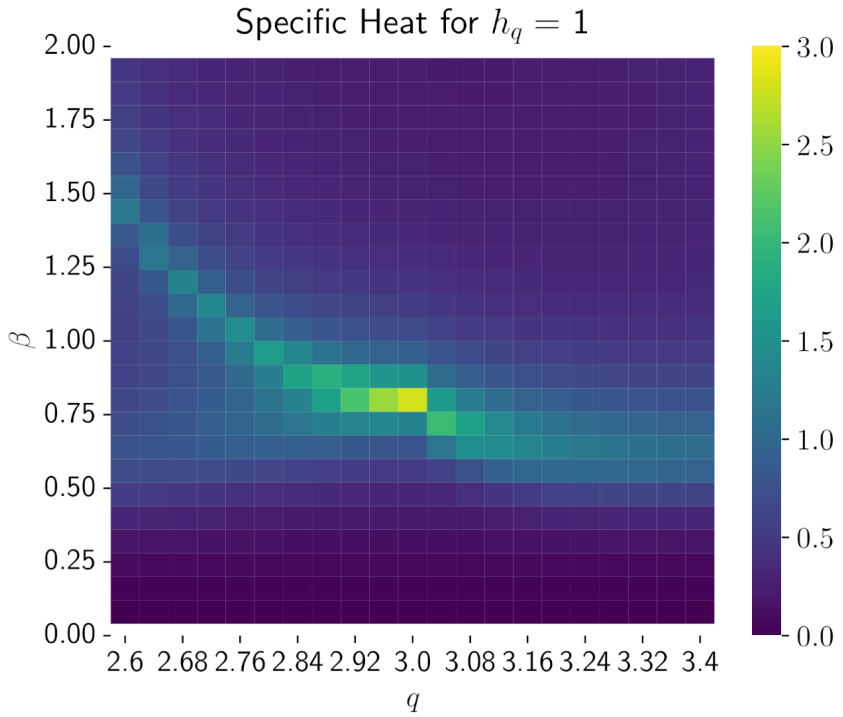
Quantum simulation of similar models with a continuously tunable parameter have been done with Rydberg atoms (Bernien et. al. Nature 551, 579-584 (2017), Keesling et. al. Nature 568, 207 (2019)). The resulting phase diagram (right) shows similarities to the proxy phase diagram of the extended-O(2) model at finite h_q.





Specific Heat from TRG with $L = 1024$ and $h_q = 1$

$$H_{\text{ext-}O(2)} = -\sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - h_q \sum_x \cos(q\varphi_x)$$



Specific Heat from TRG with $L = 1024$ and $h_q = 1$

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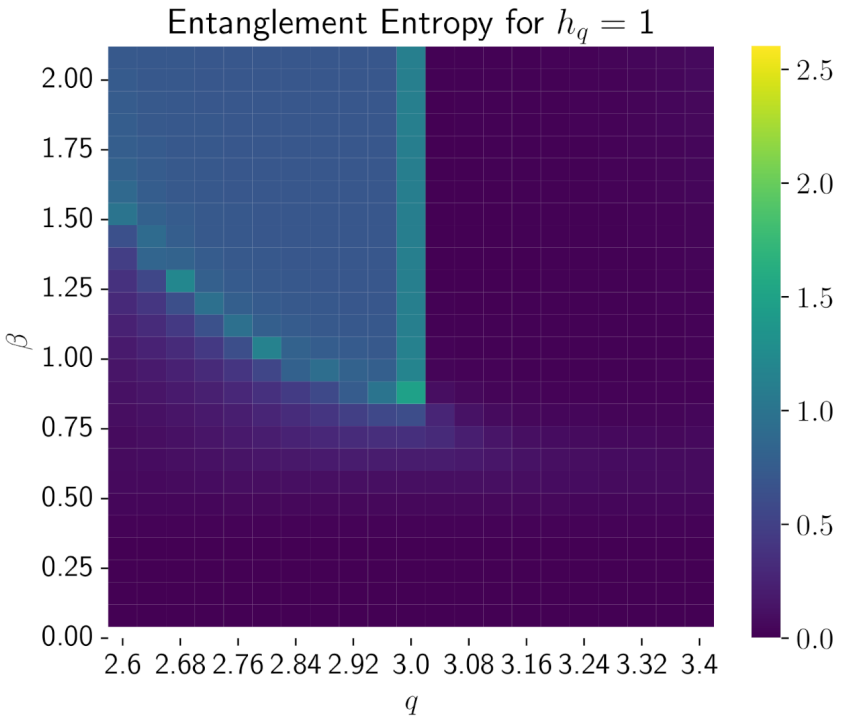
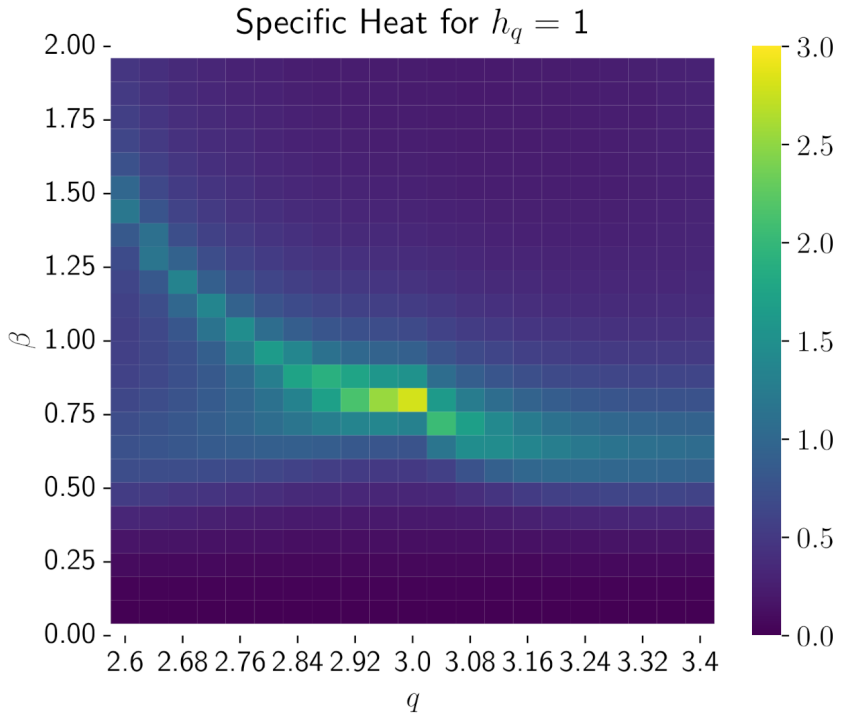
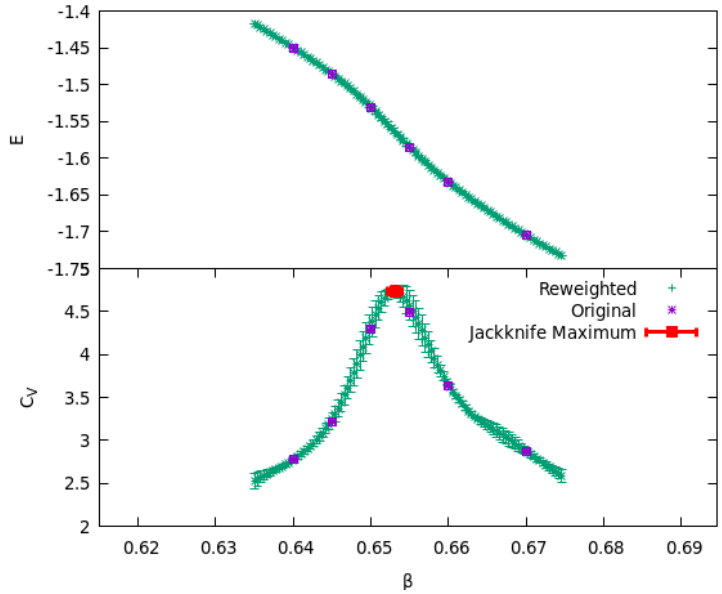
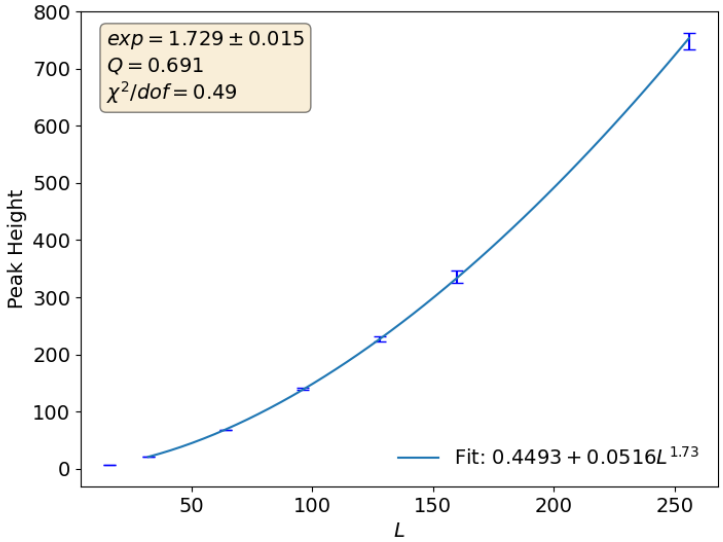




Finishing Up: Reweighting and Finite Size Scaling

$$\left.\frac{dU_M}{d\beta}\right|_{max} = U_0 + U_1 L^{1/\nu}$$
$$C_V|_{max} = C_0 + C_1 L^{\alpha/\nu}$$
$$\langle M \rangle|_{infl} = M_0 + M_1 L^{-\beta/\nu}$$
$$\chi_M|_{max} = \chi_0 + \chi_1 L^{\gamma/\nu}$$
$$F(\vec{q})|_{max} = F_0 + F_1 L^{2-\eta}.$$





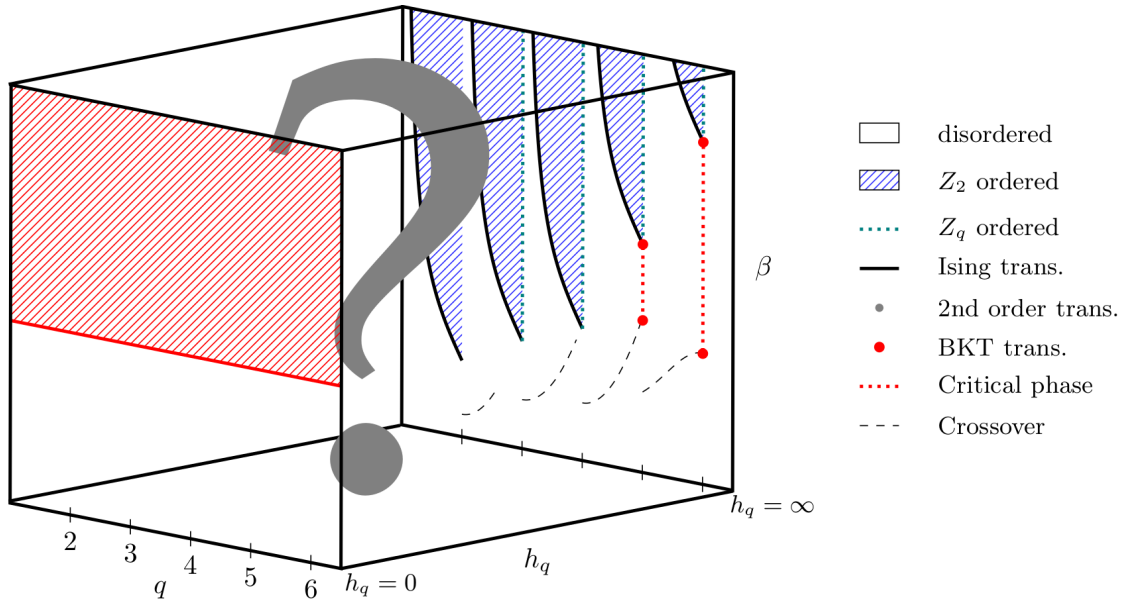
Computational Resources

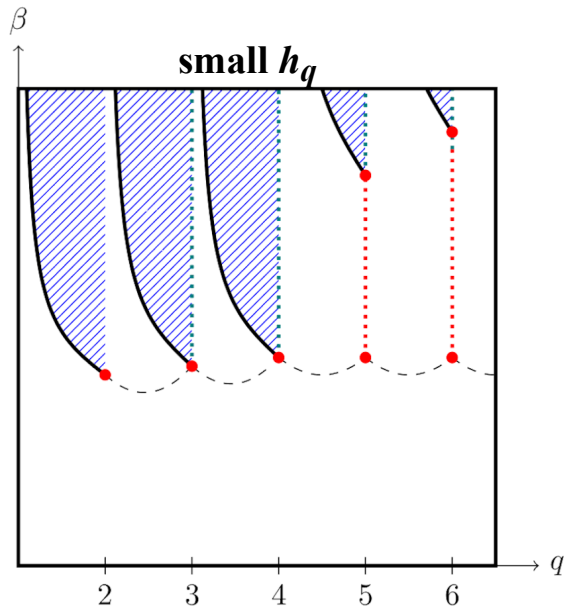
Computationally, this was a massive project and required careful workflow design with automated production and data analysis

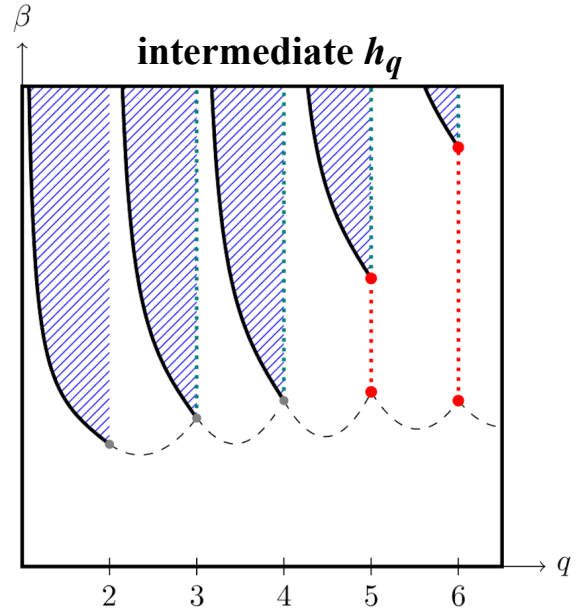
- $30K+$ Monte Carlo simulations just to perform a basic scan of the parameter space
- I ran up to 800 nodes at once (trivial parallelization) on MSU's ICER
- Large autocorrelations required Markov chains of length billions in some cases
- Several terabytes of hard disk space for the time series observables
- I used $500K+$ CPU hours on MSU's ICER

...and that's just for the MCMC.

Phase Diagram







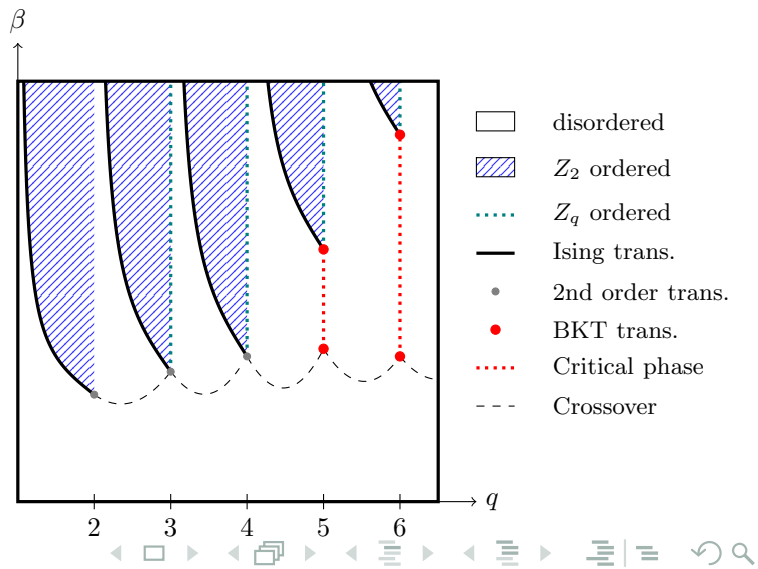
Outline

Summary

1. We looked at an extended O(2) model with parameters β, h_q, and q

$$H = -\sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - h_q \sum_x \cos(q\varphi_x)$$

2. The model is related to the Abelian-Higgs model
3. May be a good candidate for analog quantum simulation
4. The symmetry-breaking term allows us to explore the role of symmetry and to study the $U(1) \to \mathbb{Z}_q$ approximations and to consider also noninteger q
5. Rich phase diagram with crossovers, second-order phase transitions of various universality classes and BKT transitions



Acknowledgments

Committee:

- Alexei Bazavov (chair)
- Dean Lee
- Huey-Wen Lin
- Mohammad Maghrebi
- Andreas von Manteuffel

Collaborators:

- Alexei Bazavov
- Dean Lee
- Huey-Wen Lin
- Yannick Meurice
- Ryo Sakai
- Jin Zhang
- Judah Unmuth-Yockey
- Giovanni Pederiva
- Andrea Shindler
- Brandon Henke

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- TRG results computed on Syracuse University HTC Campus Grid and at NERSC[7]
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[8] Quantum Information Science Enabled Discovery (QuantISED)

Thank you!

Additional Slides:

My Papers and Conference Papers

Extended-O(2) model:

- (2023) Hostetler et. al., "Symmetry Breaking in an Extended-O(2) Model". *In progress*
- (2022) Hostetler et. al., "Symmetry Breaking in an Extended-O(2) Model". PoS(LATTICE2022)014 arXiv:2212.06893
- (2021) Hostetler et. al., "Clock model interpolation and symmetry breaking in O(2) models." Phys. Rev. D 104 054505 arXiv:2105.10450
- (2021) Meurice et. al. "From tensors to qubits". PoS(LATTICE2021)608 arXiv:2112.10005
- (2021) Hostetler et. al., "Clock model interpolation and symmetry breaking in O(2) models." PoS(LATTICE2021)353 arXiv:2110.05527

Digital quantum simulation of the Schwinger model:

- (2023) Pederiva et. al. "Quantum State Preparation for the Schwinger Model." *In progress*
- (2021) Pederiva et. al. "Quantum State Preparation for the Schwinger Model." PoS(LATTICE2021)047 arXiv:2109.11859

Lattice Abelian-Higgs Model in 1+1 D

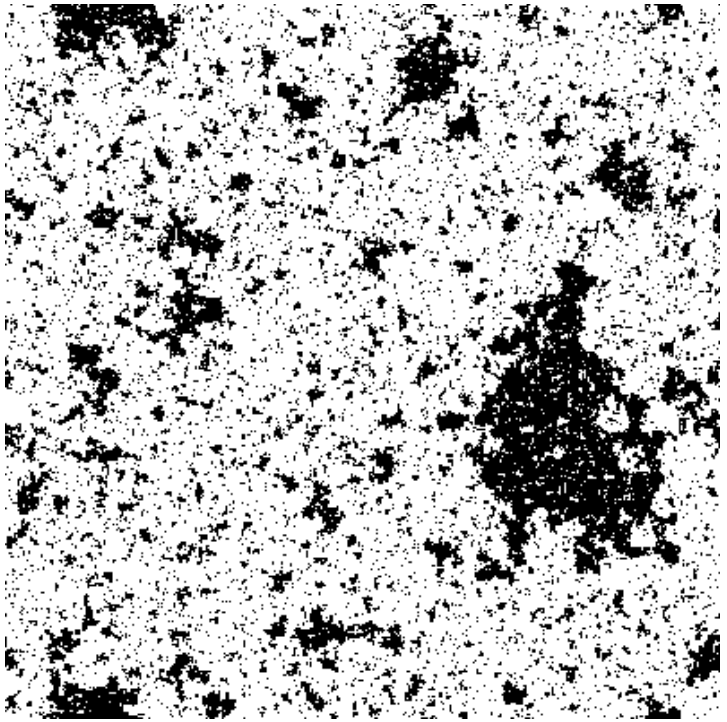
- The Schwinger model with electron replaced by complex scalar field
- The lattice action is

$$S = -\beta_{pl} \sum_x \sum_{\nu<\mu} \mathrm{Re}\left[U_{x,\mu\nu}\right] - \kappa \sum_x \sum_{\nu=1}^{2} \left[\phi_x^\dagger U_{x,\nu} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^\dagger U_{x,\nu}^\dagger \phi_x\right] + \lambda \sum_x \left(\phi_x^\dagger \phi_x - 1\right)^2 + \sum_x \phi_x^\dagger \phi_x$$

  - Scalar field $\phi_x = |\phi_x| e^{i\theta_x}$ on sites x
  - Abelian gauge fields $U_{x,\mu} = e^{iA_\mu(x)}$ on links from x to $x + \hat{\mu}$
  - Plaquettes $U_{x,\mu\nu} = e^{i[A_\mu(x) + A_\nu(x+\hat{\mu}) - A_\mu(x+\hat{\nu}) - A_\nu(x)]}$
  - Inverse gauge coupling $\beta_{pl} = 1/g^2$
  - Hopping coefficient κ
  - Scalar self-coupling λ
- Reduces to the classical O(2) model when $\lambda = \beta_{pl} = \infty$

Outline

The Ising Model



- A model of ferromagnetism
- We define discrete "atomic spins" on a lattice of N sites
- The energy of a particular configuration is

$$H = -\sum_{\langle i,j \rangle} S_i S_j - h \sum_{j=1}^{N} S_j$$

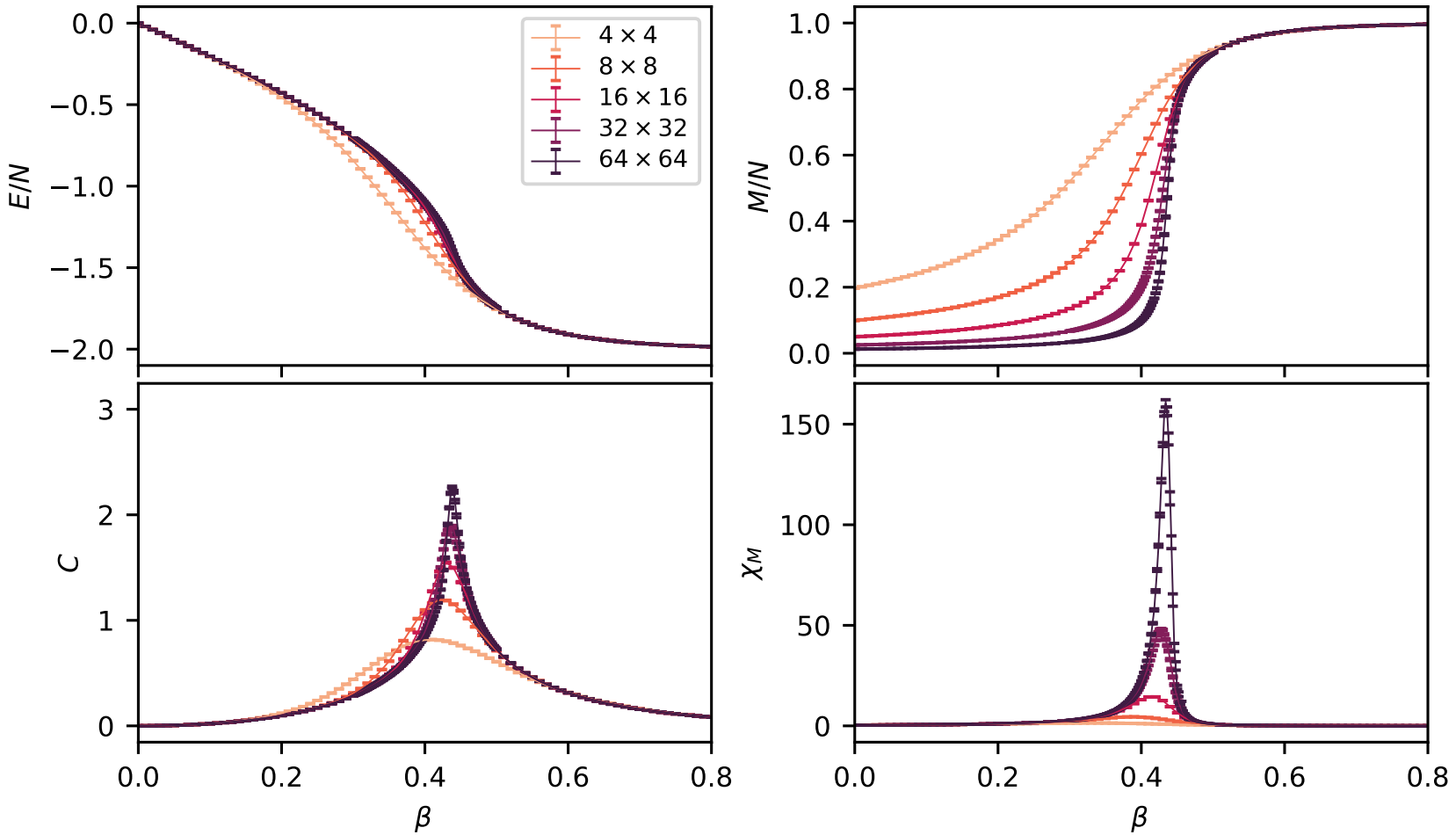
  with nearest neighbor interactions and $S_i = \pm 1$
- The partition function is

$$Z = \sum_i e^{-\beta H_i}$$

The Ising Model

- Thermodynamic functions

$$E = -\frac{\partial}{\partial \beta} \ln Z, \quad C = -\frac{\beta^2}{N} \frac{\partial E}{\partial \beta}, \quad M = \frac{1}{\beta} \frac{\partial}{\partial h} \ln Z, \quad \chi = \frac{1}{N\beta} \frac{\partial M}{\partial h}$$



Markov Chain Monte Carlo (MCMC)

- The equilibrium expectation value of an observable O is

$$\langle O \rangle = \sum_i O_i P_i, \qquad P_i = \frac{e^{-\beta E_i}}{Z}$$

- Instead of direct enumeration or naive Monte Carlo sampling, we must do *importance sampling*
- If N microstates are selected according to the equilibrium distribution $P_i = e^{-\beta E_i}/Z$, then

$$\langle O \rangle \approx \frac{1}{N} \sum_{i=1}^{N} O_i$$

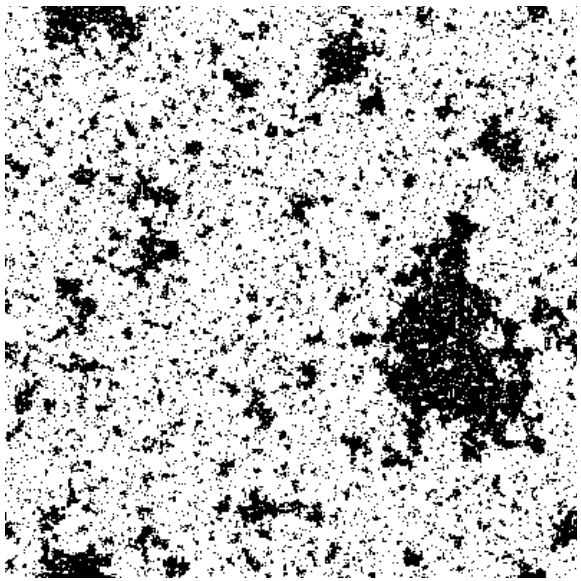
- Start with some arbitrary microstate U_0 and construct a Markov chain (via e.g. Metropolis algorithm)

$$U_0 \xrightarrow{update} U_1 \xrightarrow{update} U_2 \xrightarrow{update} \cdots$$

such that the chain eventually reaches the equilibrium distribution P_i

Classical Spin Systems in 2D

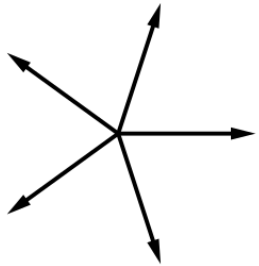
$$H = -J \sum_{x,\mu} \vec{S}_x \cdot \vec{S}_{x+\hat{\mu}} = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)$$

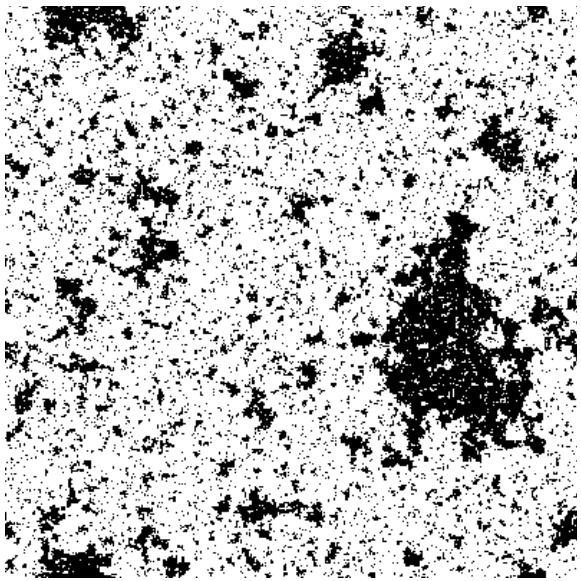


(a) Ising Model

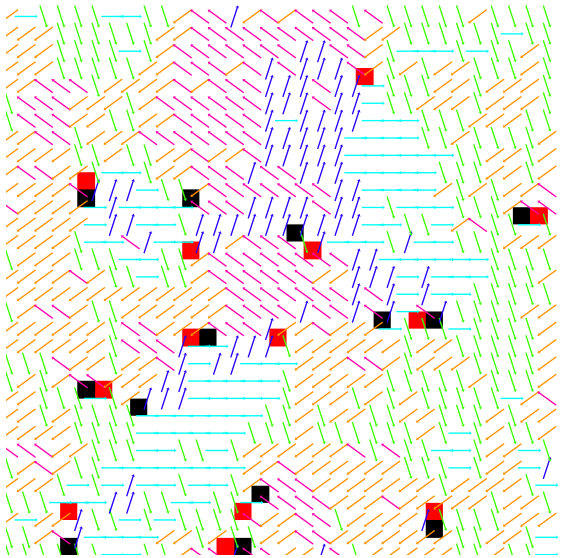
Classical Spin Systems in 2D

$$H = -J \sum_{x,\mu} \vec{S}_x \cdot \vec{S}_{x+\hat{\mu}} = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)$$





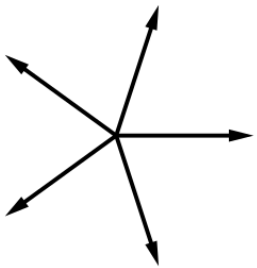
(a) Ising Model

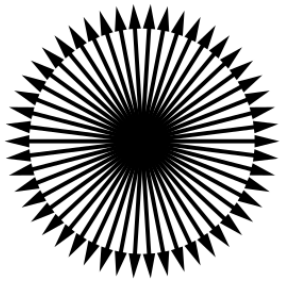


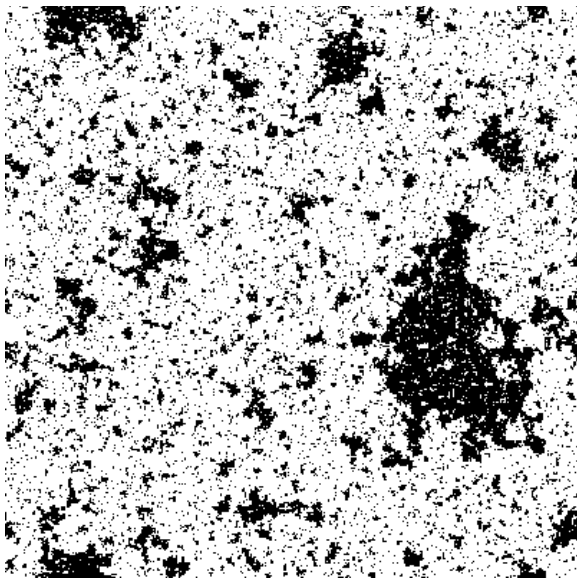
(b) Clock Models

Classical Spin Systems in 2D

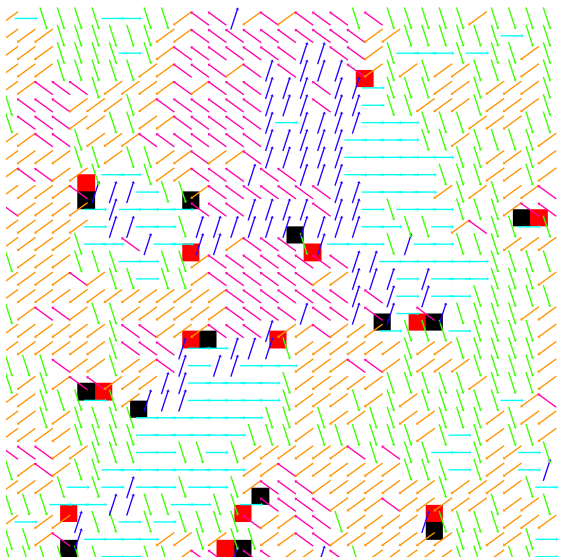
$$H = -J \sum_{x,\mu} \vec{S}_x \cdot \vec{S}_{x+\hat{\mu}} = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)$$



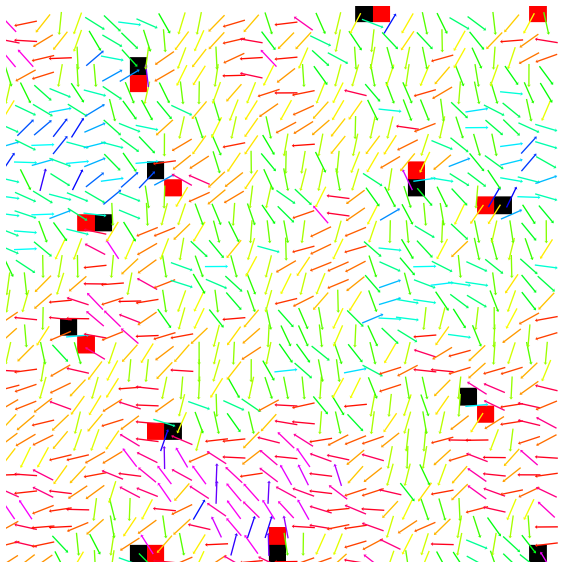




(a) Ising Model



(b) Clock Models



(c) XY Model

Previous Work

- José, Kadanoff, Kirkpatrick, and Nelson, Phys. Rev. B 16, 1217 (1977).
- Landau, Journal of Magnetism and Magnetic Materials 31-34, 1115 (1983)
- Hu and Ying, Physica A: Statistical Mechanics and its Applications 140, 585 (1987)
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- Calabrese and Celi, Phys. Rev. B 66, 184410 (2002)
- Rastelli, Regina, and Tassi, Phys. Rev. B 69, 174407 (2004)
- Rastelli, Regina, and Tassi, Phys. Rev. B 70, 174447 (2004)
- Taroni, Bramwell, and Holdsworth, Journal of Physics: Condensed Matter 20, 275233 (2008)
- Nguyen and Ngo, Advances in Natural Sciences: Nanoscience and Nanotechnology 8, 015013 (2017)
- Chlebicki and Jakubczyk, Phys. Rev. E 100, 052106 (2019)
- Butt, Jin, Osborn, and Saleem, (2022), arXiv:2205.03548

Markov Chain Monte Carlo (MCMC)

Need an updating algorithm that obeys:

- Every microstate must be reachable (ergodicity)
- The Markov process must eventually reach the equilibrium distribution P_i and stay there

Metropolis algorithm:

1. Given the current configuration U_t, generate a candidate configuration U' by some random process
2. Accept this candidate as the new configuration U_{t+1} with probability

$$P_A = \min\left(1, e^{-\beta \Delta E}\right)$$

3. Repeat these steps

MCMC for Quantum Field Theories

- Vacuum expectation values are path integrals

$$\langle O \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\ O[A, \overline{\psi}, \psi]\ e^{iS[A,\overline{\psi},\psi]}}{\int \mathcal{D}A\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\ e^{iS[A,\overline{\psi},\psi]}}$$

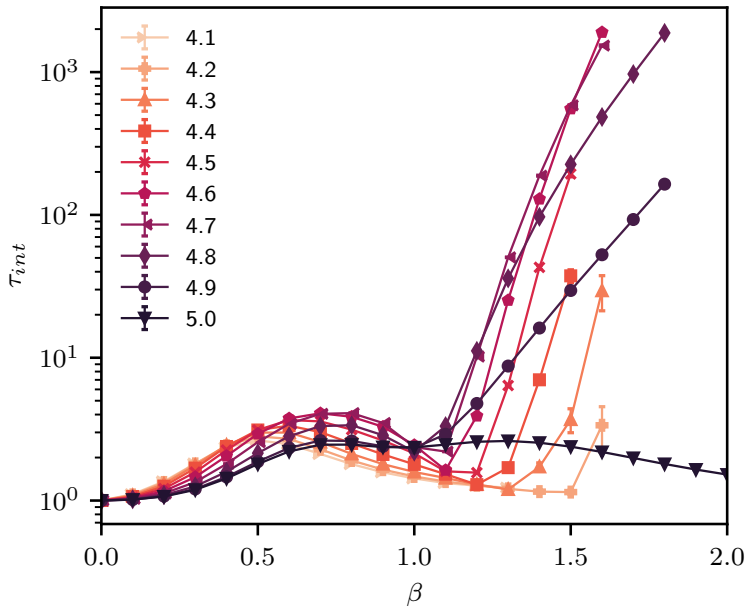
- After lattice regularization and Wick rotation ($t \to it$)

$$\langle O \rangle = \frac{\int \mathcal{D}U\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\ O[U, \overline{\psi}, \psi]\ e^{-S_E[U,\overline{\psi},\psi]}}{\int \mathcal{D}U\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\ e^{-S_E[U,\overline{\psi},\psi]}}$$

- Limited to equilibrium physics
- For dynamical physics, need a new approach e.g. quantum simulation

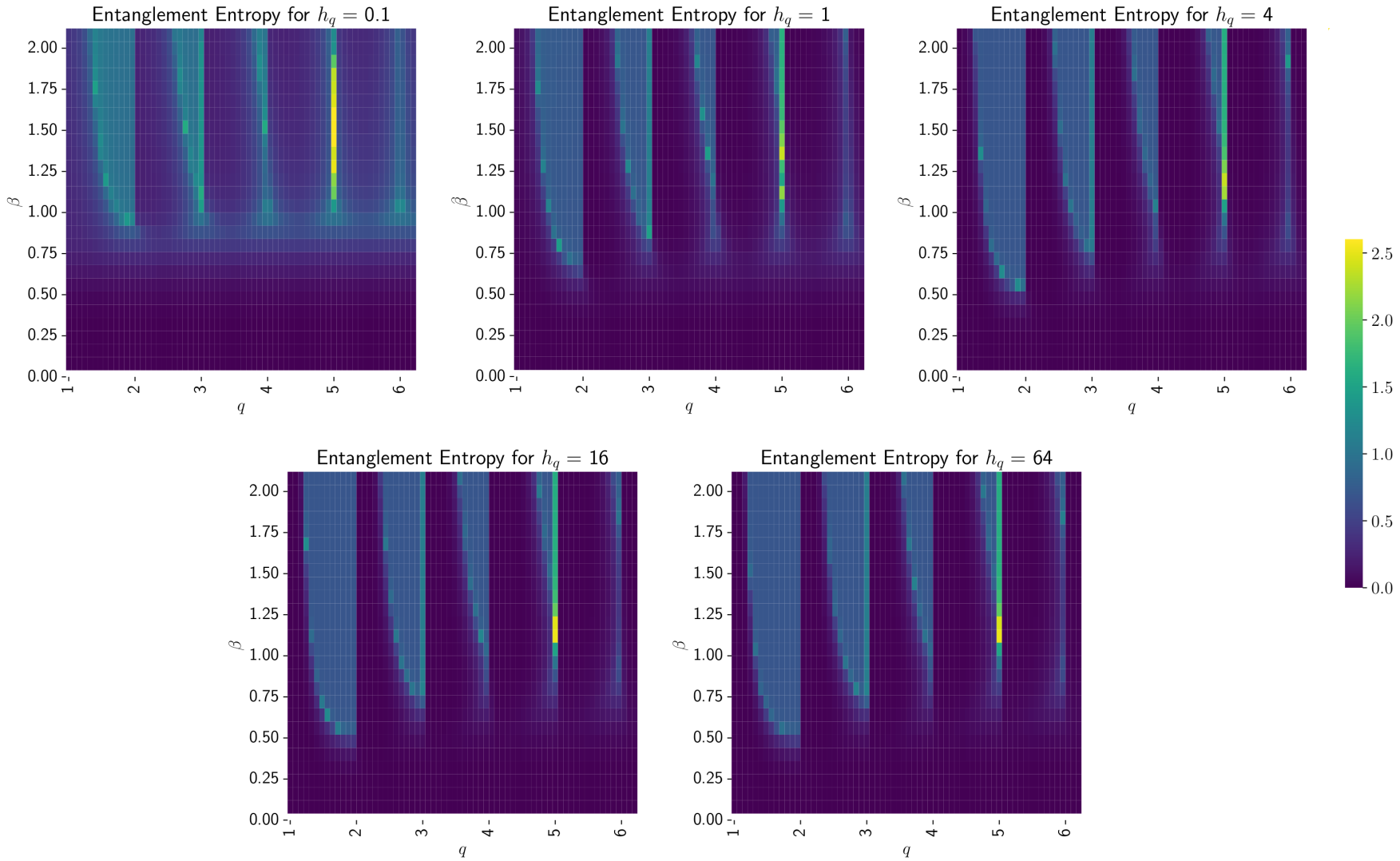
TRG

- In the Monte Carlo approach, we use a Markov chain importance-sampling algorithm to generate equilibrium configurations
  - Monte Carlo has difficulty sampling this model appropriately at $\beta > 1$ for $q \notin \mathbb{Z}$
  - Integrated autocorrelation time explodes, and we have to perform billions of heatbath sweeps already on a 4×4 lattice
  - Studying this model on larger lattices with Monte Carlo is challenging

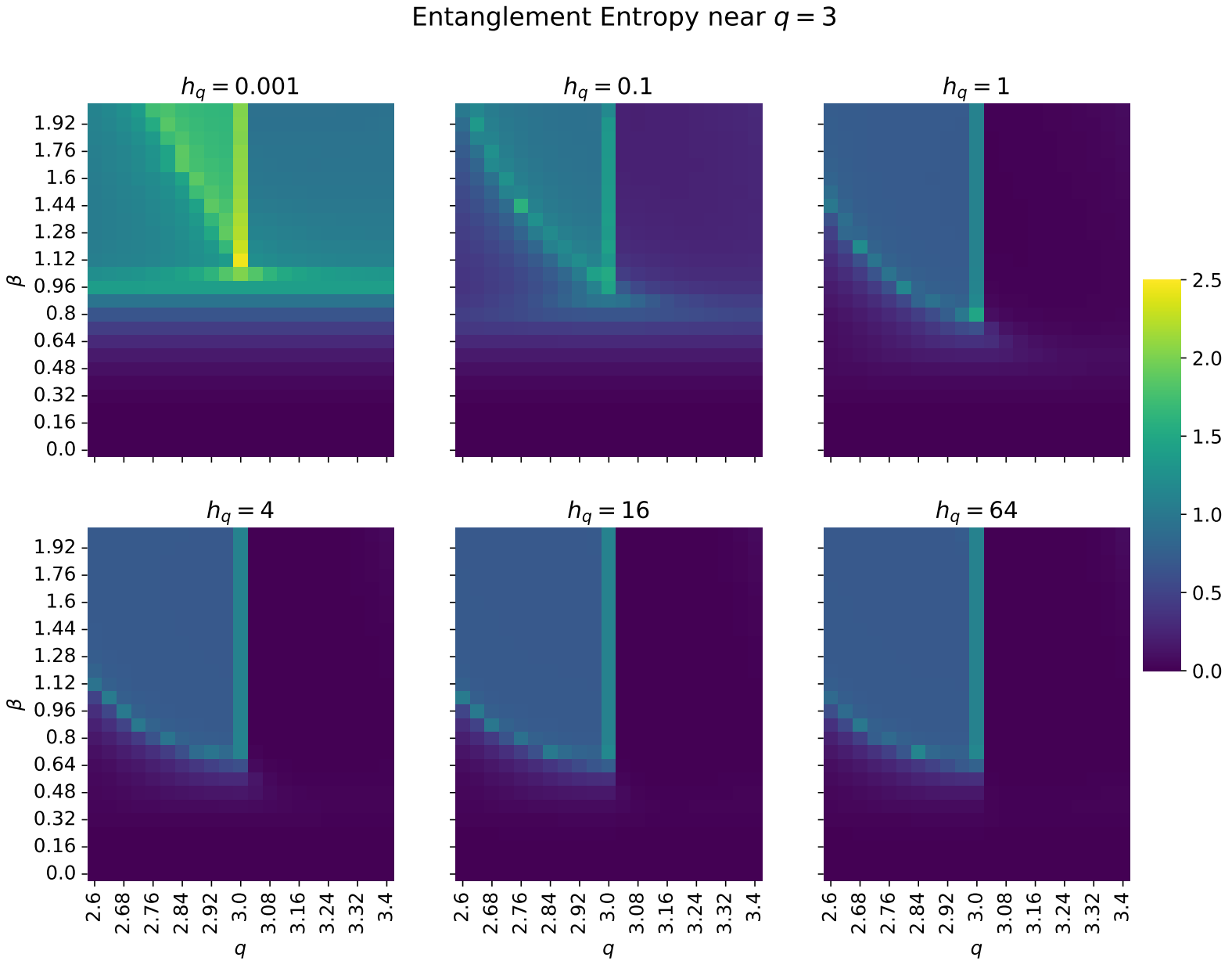


- Tensor renormalization group (TRG) approach can be used instead
  - We validate TRG against Monte Carlo in the regime accessible to Monte Carlo
  - Then we use TRG to explore lattice sizes and β-values beyond the reach of Monte Carlo

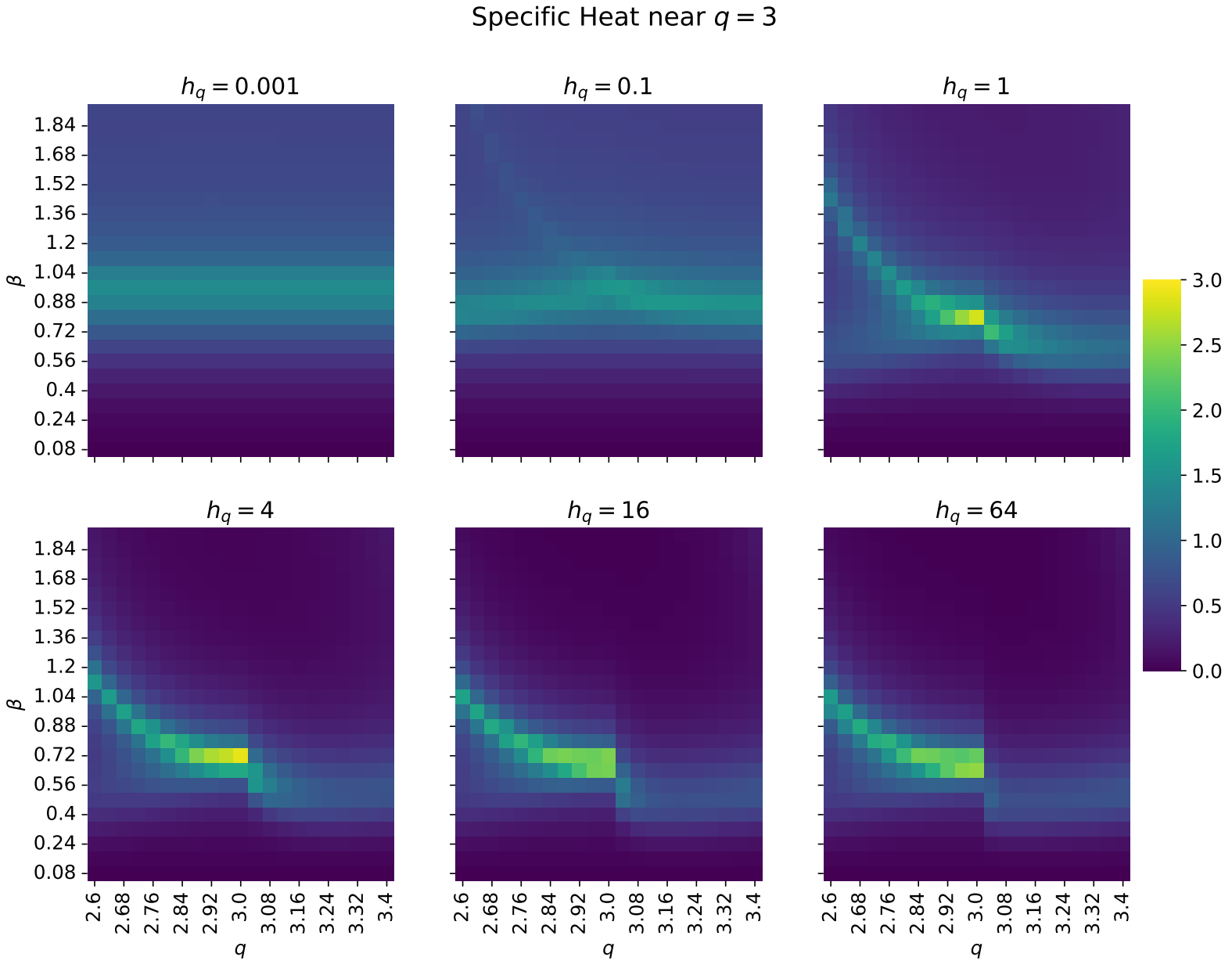
Entanglement Entropy from TRG with $L = 1024$



Entanglement Entropy from TRG with $L = 1024$

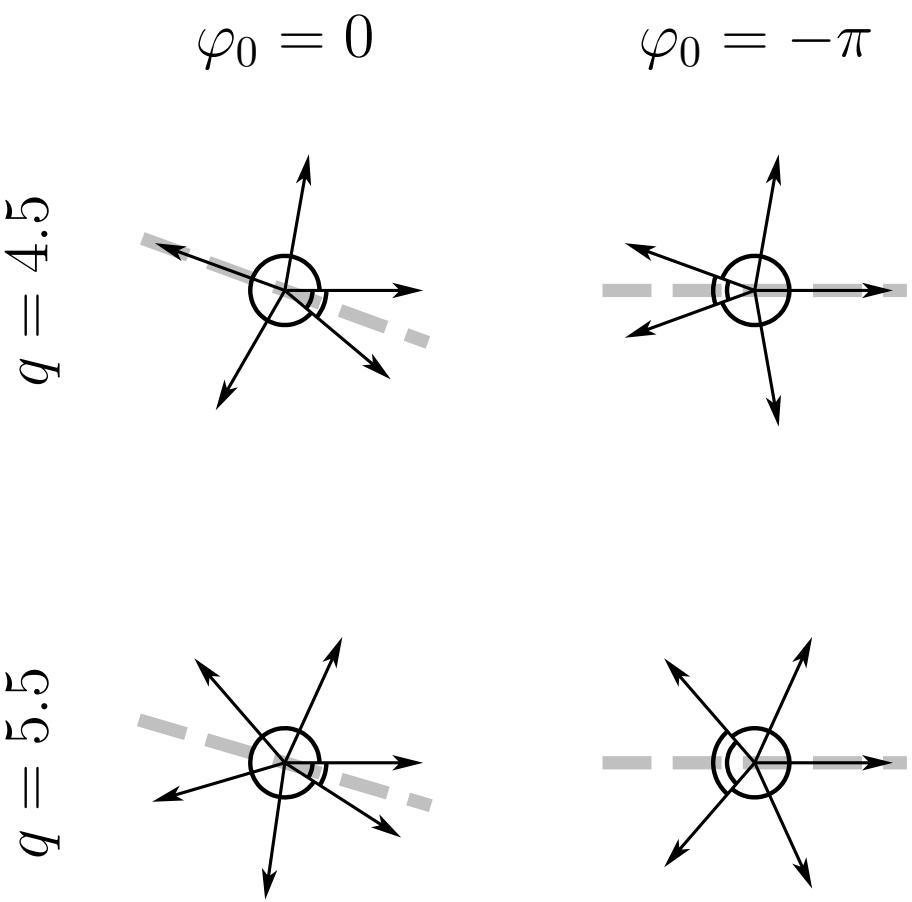


Specific Heat from TRG with $L = 1024$

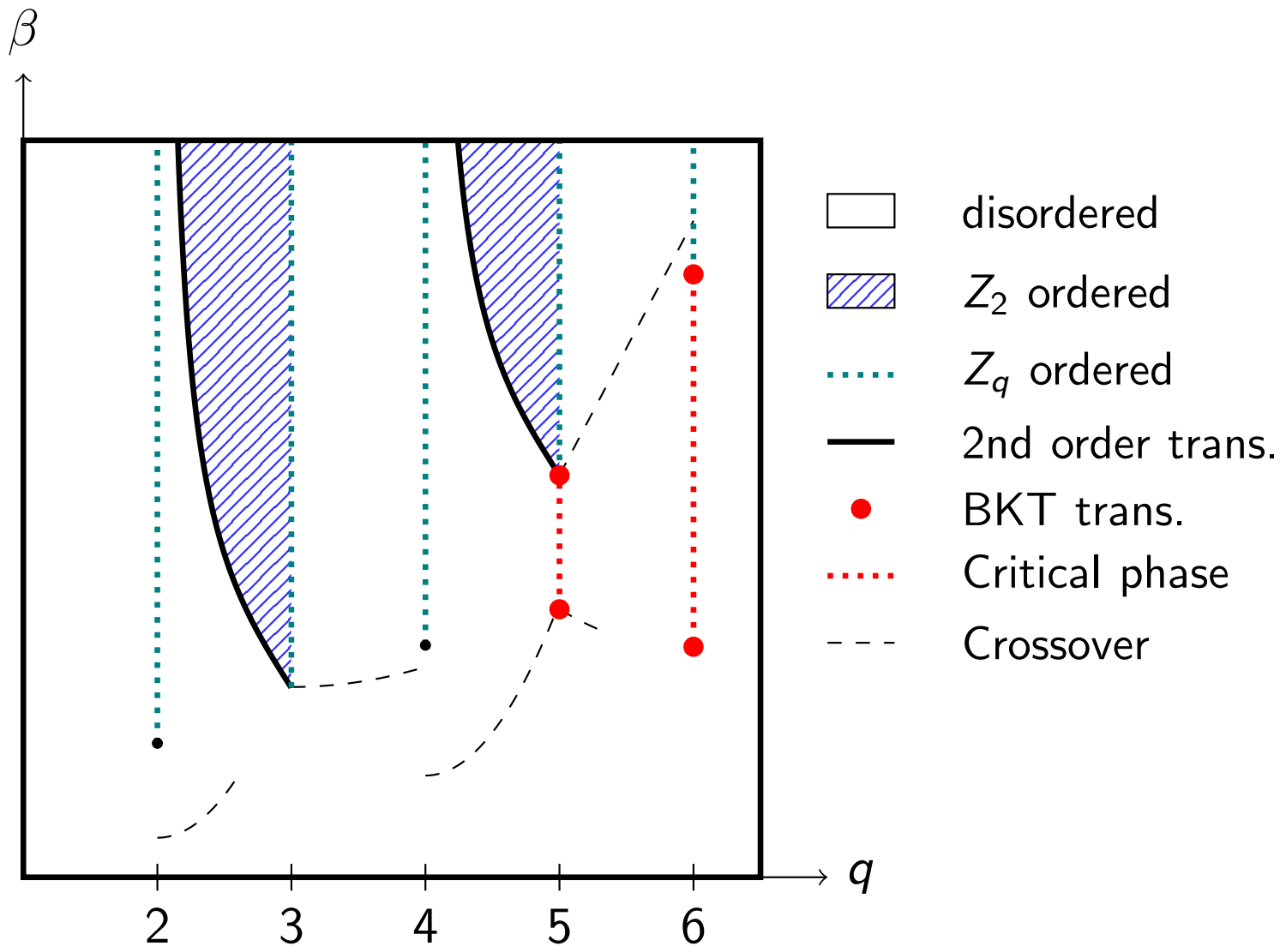


Choice of φ_0

- Choice of φ_0 can change the DOF in the model
- We choose $\varphi_0 = 0$, i.e. $\varphi \in [0, 2\pi)$, but we also investigate $\varphi_0 = -\pi$



Phase diagram for $h_q = \infty$ and $\varphi_0 = -\pi$



Placement of β

- One can define the model as

$$H = -\beta \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - h_q \sum_x \cos(q\varphi_x)$$

  where β is multiplying the first term like a field-theoretic coupling. Then the Boltzmann factor is e^{-S}
- Alternatively, one can factor β out front and define the model as

$$H = -\sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - h'_q \sum_x \cos(q\varphi_x)$$

  with Boltzmann factor $e^{-\beta S}$, where β is the inverse temperature
- The two definitions are related by $h'_q = h_q/\beta$
- We have used both definitions, however, the Monte Carlo results shown in these slides are from the definition with β factored out front

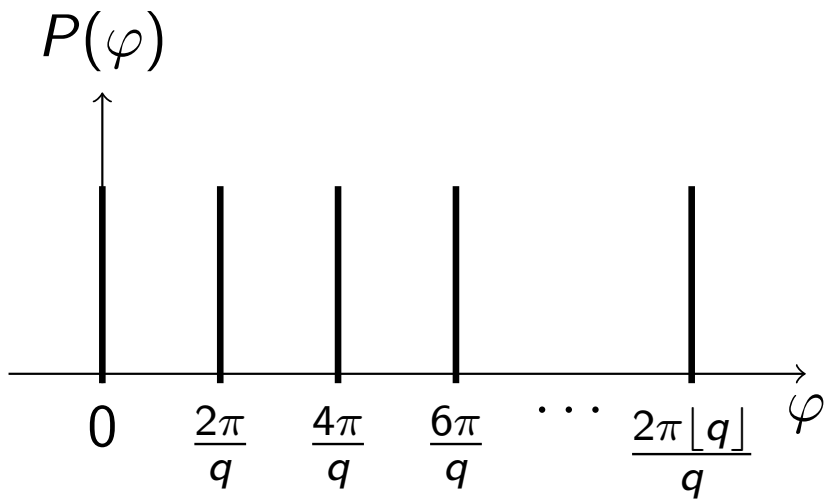
The Need to Shift the Angles: A Subtlety

- In the ordinary clock model, we have the energy function

$$H = -\sum_{\langle x,y \rangle} \cos(\varphi_x - \varphi_y)$$

- The angles $\varphi_x^{(k)}$ are selected discretely as $\varphi_0 \leq \varphi_x^{(k)} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$
- When $\beta = 0$ and with $\varphi_0 = 0$, the spins are selected uniformly from a "Dirac comb"

$$P^{clock}_{q,\varphi_0=0}(\varphi) \sim \sum_{k=0}^{\lfloor q \rfloor} \delta\left(\varphi - \frac{2\pi k}{q}\right)$$



The Need to Shift the Angles: A Subtlety

- In the Extended-O(2) model, we have the energy function

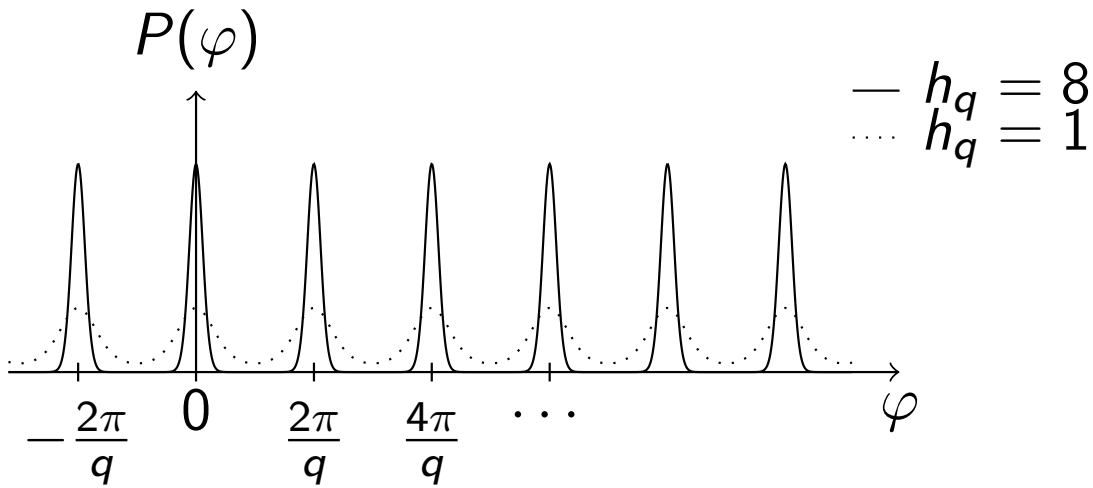
$$H = -\sum_{\langle x,y \rangle} \cos(\varphi_x - \varphi_y) - h_q \sum_x \cos(q\varphi_x)$$

- The angles φ_x are now selected continuously in

$$\varphi_0 \leq \varphi \in \mathbb{R} < \varphi_0 + 2\pi$$

- When $\beta = 0$ and with $\varphi_0 = 0$, the spins are selected from a distribution

$$P^{extO2}_{q,\varphi_0}(\varphi) \sim e^{h_q \cos(q\varphi)}$$



The Need to Shift the Angles: A Subtlety

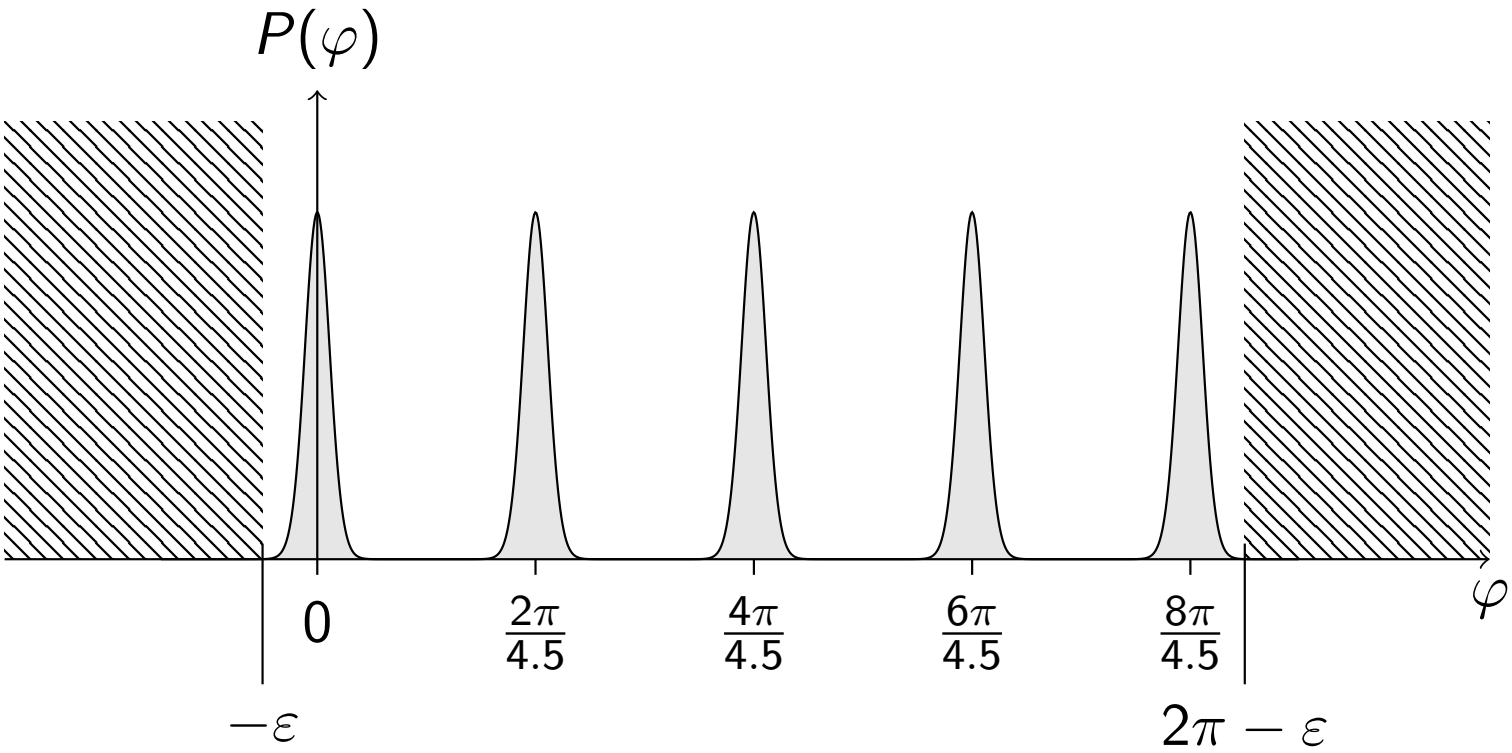


Figure: To recover the Dirac comb of the clock model distribution in the $h_q \to \infty$ limit, the angle domain must be shifted by some ε so that the histogram includes all relevant peaks.

The Need to Shift the Angles: A Subtlety

- To match the clock model in the $h_q \to \infty$ limit, it should be sufficient to choose ε such that

$$P^{extO2}_{q,\varphi_0}(\varphi) \xrightarrow[h_q \to \infty]{} P^{clock}_{q,\varphi_0}(\varphi)$$

  where for the clock model, angles are selected from $[\varphi_0, \varphi_0 + 2\pi)$, but for the Extended-O(2) model, they are selected from $[\varphi_0 - \varepsilon, \varphi_0 - \varepsilon + 2\pi)$

- In our case, we use $\varphi_0 = 0$, and choose

$$\varepsilon = \pi \left(1 - \frac{\lfloor q \rfloor}{q}\right)$$

  so that the $\lceil q \rceil$ peaks of the distribution $P^{extO2}_{q,\varphi_0}(\varphi)$ are centered in the domain $[-\varepsilon, 2\pi - \varepsilon)$