Some Different Ways to Sum a Series

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Outline

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Euler (1735)

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The Basel Problem

In 1644, Pietro Mengoli posed the famous Basel problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$$

Euler's Solution

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Dividing both sides by x gives us

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Euler's Solution

Factoring this into its zeros, like we would a polynomial, gives us

$$\frac{\sin x}{x} = \left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{2\pi}\right)\cdots$$

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Using the difference of two squares, we get

$$\frac{\sin x}{x} = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\cdots$$

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Multiplying it out gives us

$$\frac{\sin x}{x} = 1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)x^2 + O(x^4)+\cdots$$

Euler's Solution

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Euler equates the coefficients of x^2 and concludes that

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)$$

Euler's Solution

$$-\frac{1}{3!} \;=\; -\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right)$$

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$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Bridger and Zelevinsky (2004)

Recall that the generalized Fourier series for $f(x)$ over $[-\pi, \pi]$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

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Definition: The *inner product* of two functions $f(x)$ and $g(x)$ defined on $(-L, L)$ is given by

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Now, we take the inner product of both sides of the equation

$$x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin(nx).$$

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The left side gives us

$$\langle x, x\rangle = \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^3}{3}.$$

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Setting the two results equal to each other, we get

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which simplifies to

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We start with the improper integral

$$I = \int_0^{\frac{\pi}{2}} \log(2\cos x)\, dx.$$

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$$I = \frac{i\pi^2}{8} + \int_0^{\frac{\pi}{2}} \log(1 + e^{-2ix})\, dx$$

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Rewriting the integral as a power series, we get

$$I = \frac{i\pi^2}{8} + \int_0^{\frac{\pi}{2}} \left(e^{-2ix} - \frac{e^{-4ix}}{2} + \frac{e^{-6ix}}{3} - \frac{e^{-8ix}}{4} + \cdots \right) dx.$$

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Integrating term by term, gives us

$$I = \frac{i\pi^2}{8} - \frac{1}{2i} \left(\frac{e^{-2ix}}{1} - \frac{e^{-4ix}}{2^2} + \frac{e^{-6ix}}{3^2} - \frac{e^{-8ix}}{4^2} + \cdots \right) \Big|_0^{\frac{\pi}{2}}.$$

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Evaluating from 0 to $\frac{\pi}{2}$ gives us

$$I = \frac{i\pi^2}{8} - \frac{1}{2i} \left(e^{-i\pi} - 1 - \frac{e^{-2i\pi} - 1}{2^2} + \frac{e^{-3i\pi} - 1}{3^2} - \frac{e^{-4i\pi} - 1}{4^2} + \cdots \right).$$

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Simplifying, we get

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Because one-fourth of the terms in the $p = 2$ series are even, the remaining three-fourth must be odd

$$\sum_{n\geq 0} \frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n\geq 1} \frac{1}{n^2}.$$

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Returning to our integral, we have

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$$\sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}$$

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We start with the geometric series

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$$\frac{1}{1-xy}=\sum_{n=0}^{\infty}(xy)^n \qquad\qquad \text{if } |xy|<1$$

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy \;=\; \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy$$

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$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy &= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n\, dx\, dy \\
&= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n\, dx\, dy
\end{aligned}$$

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Doing the integration, we find that

$$\sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n \, dx \, dy \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^1 y^n \, dy$$

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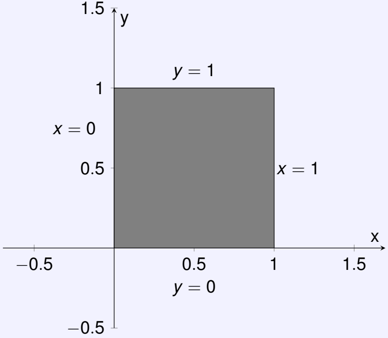
Now, we shift our attention to

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy.$$

Leveque (1956)



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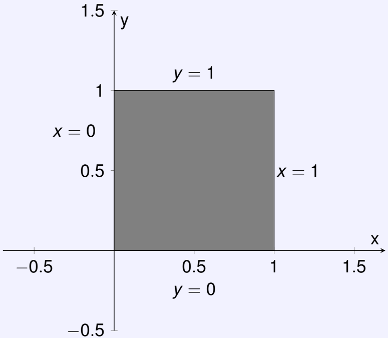

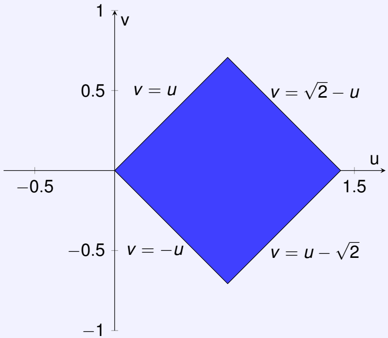

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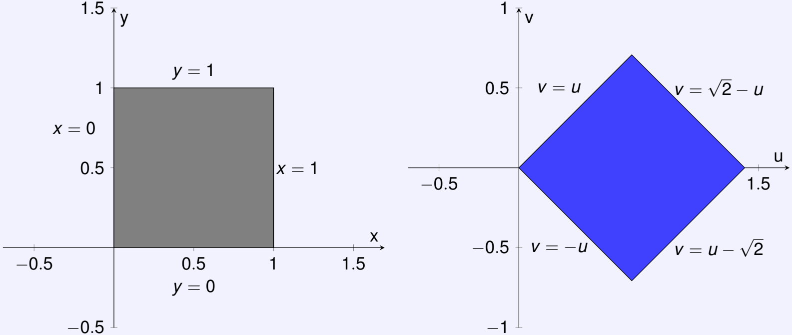

This gives us the substitutions $x = \frac{u-v}{\sqrt{2}}$ and $y = \frac{u+v}{\sqrt{2}}$.

Leveque (1956)

$$\int_0^{\frac{1}{\sqrt{2}}} \int_{-u}^{u} \frac{2}{2-u^2+v^2}\, dv\, du + \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} \int_{u-\sqrt{2}}^{\sqrt{2}-u} \frac{2}{2-u^2+v^2}\, dv\, du$$

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Letting $a^2 = 2 - u^2$, this expression becomes

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Letting $a^2 = 2 - u^2$, this expression becomes

$$2\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2-u^2}} \tan^{-1}\left(\frac{v}{\sqrt{2-u^2}}\right)\Big|_{-u}^{u} du$$

$$+2\int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \tan^{-1}\left(\frac{v}{\sqrt{2-u^2}}\right)\Big|_{u-\sqrt{2}}^{\sqrt{2}-u} du$$

Leveque (1956)

$$4\int_0^{\frac{\pi}{6}} \theta\, d\theta + 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta$$

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$$4\int_0^{\frac{\pi}{6}} \theta\, d\theta + 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta = \frac{\pi^2}{6}$$

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In summary...

$$\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\, dx\, dy \quad = \quad \int_0^1\int_0^1 \frac{1}{1-xy}\, dx\, dy$$

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$$\frac{1}{\left[2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)\right]^2}$$

$$\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\cos^2\left(\frac{x}{2}\right)}\right]$$

Chen (2010)

Substitute $\cos^2\left(\frac{x}{2}\right)$ with $\sin^2\left(\frac{\pi}{2}+\frac{x}{2}\right)$

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$$1 = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)}+\frac{1}{\sin^2\left(\frac{3\pi}{4}\right)}\right]$$

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Chen (2010)

Repeating this process gives us

$$1 = \frac{2}{16}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)}\right].$$

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Chen (2010)

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Tannery's Theorem: If $\lim_{n\to\infty} f_m(n) = f_m$, then

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$$\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$$

Thank you for your attention!

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