Clock model interpolation and symmetry breaking in O(2) models

Leon Hostetler [1], Ryo Sakai [2], Jin Zhang [2],
Judah Unmuth-Yockey [3], Alexei Bazavov [1], and Yannick Meurice [2]

[1] Michigan State University

[2] University of Iowa

[3] Fermilab

November 2, 2020
QuLAT Collaboration

Outline

1. The q-state clock model
   - Energy density and specific heat
   - Phase diagram
2. The gamma model
3. The toy clock model with fractional q
   - Energy density and specific heat
   - Validating TRG
   - TRG results at large volume and large β
   - Large and small β limits
   - Summary

Motivation

- One challenge of mapping quantum field theories to quantum simulators is dealing with the infinite Hilbert spaces
- One possibility:
  1. Truncate the "spin" states
  2. Map the system to a quantum simulator. See for example arXiv: 1403.5238, 1503.08354, and 1803.11166
  3. Tensor renormalization group (TRG) methods can be used to study the effect of these truncations
- Or from the other direction:
  1. Start with a model that is already truncated, such as the classical q-state clock model
  2. Construct a quantum Hamiltonian and take time-continuum limit
  3. Map the system to a quantum simulator
  4. Learn from this to help us move on to more challenging models

The q-state clock model

- For the q-state clock model[1], the energy function is

$$E = -J \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right)$$

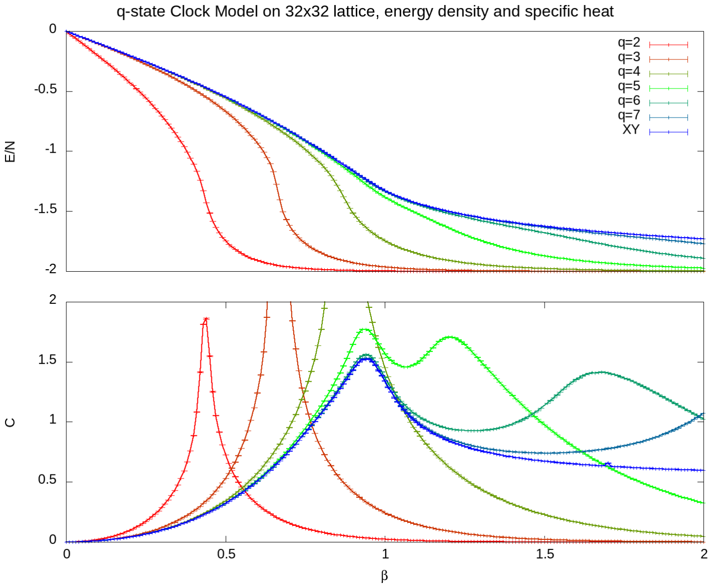
- The spins reside on lattice sites and take on values

$$\varphi \in \left\{0, \frac{2\pi}{q}, \frac{4\pi}{q}, \cdots, \frac{2\pi(q-1)}{q}\right\}$$

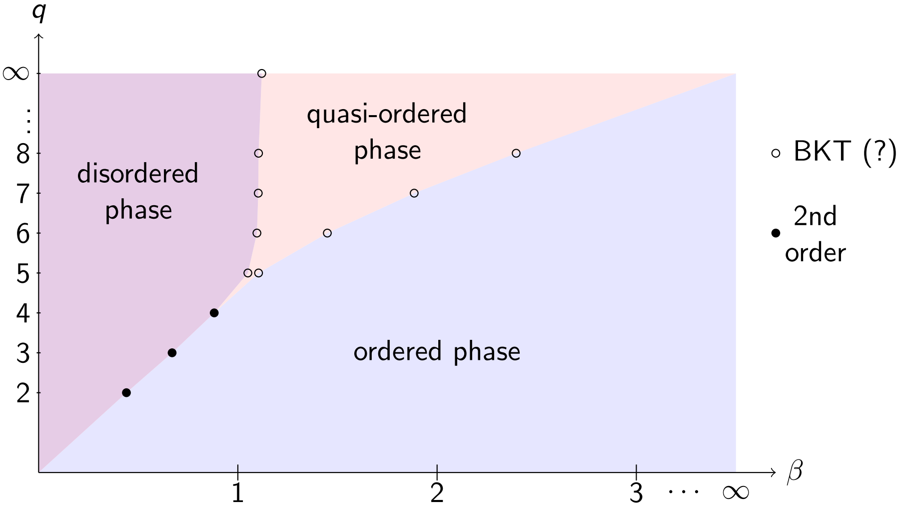
- This model has a discrete $\mathbb{Z}_q$ symmetry and interpolates between the Ising and XY models
  1. For $q = 2$, it is equivalent to Ising model
  2. For $q = 3$, it is equivalent to the *standard* 3-state Potts model
  3. For $q = 4$, it is equivalent to two Ising models
  4. For $5 \leq q < \infty$, it has two phase transitions
  5. For $q \to \infty$, it becomes the continuous XY model

[1] Also called the "planar Potts model", the "vector Potts model", or the "$\mathbb{Z}_q$ model".

Clock model energy density and specific heat



Clock model phase diagram



Overview

The gamma model

- We consider now the energy function

$$E = -J \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - \gamma \sum_x \cos\left(q\varphi_x\right)$$

- The spins reside on lattice sites and take on values

$$\varphi \in [0, 2\pi)$$

- Note:
  - When $\gamma = 0$, this is the XY model
  - When $\gamma \to \infty$ and $q \in \mathbb{Z}$, this becomes the clock model
- By slowly turning on γ, we can explore the effect of breaking the O(2) symmetry
- We no longer require $q \in \mathbb{Z}$, so this allows us to consider the "clock model" with fractional q
- We haven't studied this specific form of the model yet. We have studied a very similar model with $q\Delta\varphi$ in the second cosine. Those results are in the appendix if you're interested

Overview

The toy clock model with fractional q

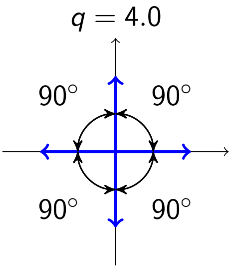
- We return to the clock model

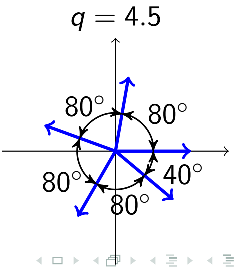
$$E = -J \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right)$$

- But now q is not necessarily an integer and the spins take on the values

$$\varphi \in \left\{0, \frac{2\pi}{q}, \frac{4\pi}{q}, \cdots, \frac{2\pi\lfloor q \rfloor}{q}\right\}$$

- When $q \in \mathbb{Z}$, the result is the ordinary q-state clock model with $\mathbb{Z}_q$ symmetry. When $q \notin \mathbb{Z}$, the $\mathbb{Z}_q$ symmetry is broken. For example:



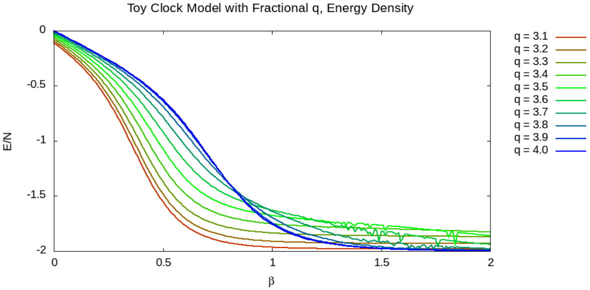


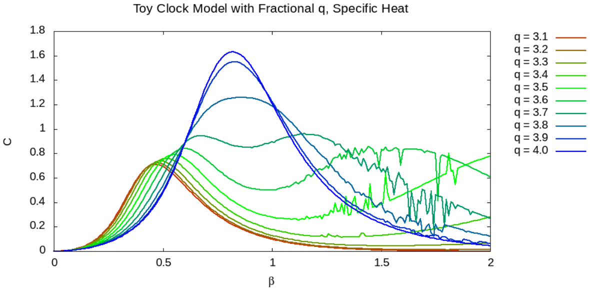
Monte Carlo Results

- Monte Carlo approach is well understood and reliable
- However, for this model, large β and large volumes are computationally intensive
- On the next few slides we show some quick (i.e. low statistics) results done on a small 4x4 lattice
- Error bars are not shown and the results show some fluctuation at large β due to insufficient statistics, but the curves give a good qualitative picture[2]. Think of these as *preliminary* plots
- Later, we harness TRG (tensor renormalization group) to explore large β and large volume

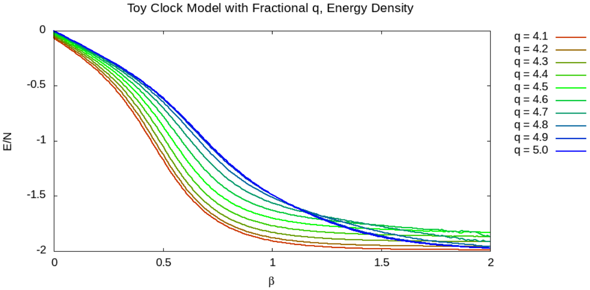
[2] We've verified this by performing a number of high statistics runs with error analysis

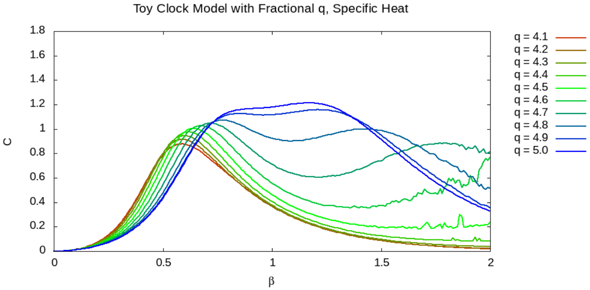
Toy Model: $3.1 \leq q \leq 4.0$



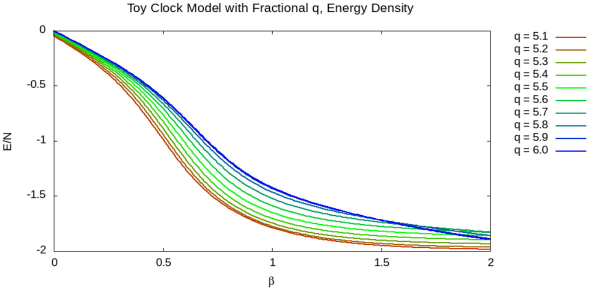


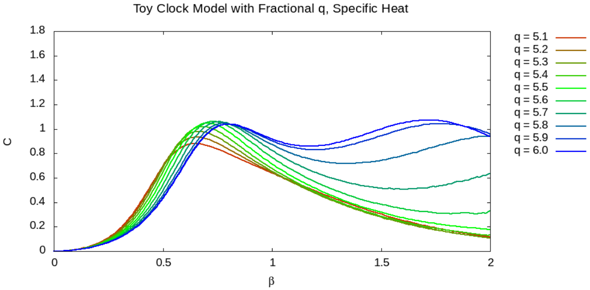
Toy Model: $4.1 \leq q \leq 5.0$





Toy Model: $5.1 \le q \le 6.0$

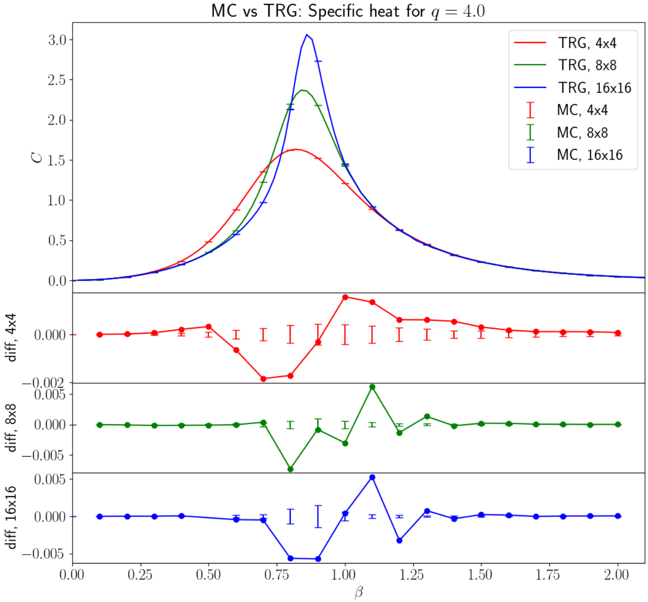




TRG

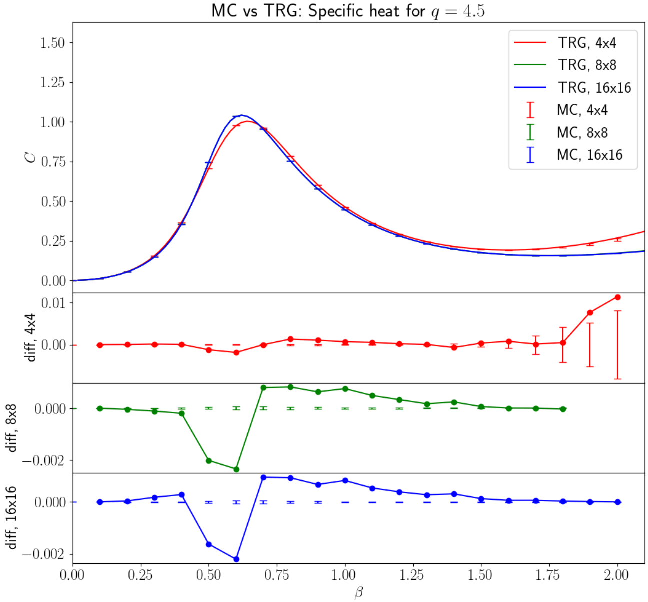
- Ryo Sakai is able to study this toy model using tensor renormalization group (TRG) approach
- The benefit over Monte Carlo (MC) is that TRG can go to much larger β and much larger volume
- We start by validating TRG by comparing it with MC at small β and small volumes
- Then we show some of the large β and large volume results

Validating TRG: $q = 4.0$

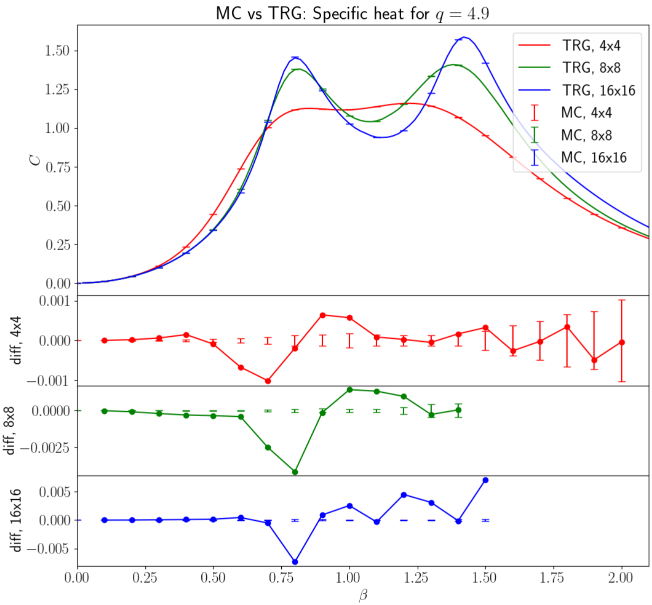


← Deviation is due to discretization error from the derivative

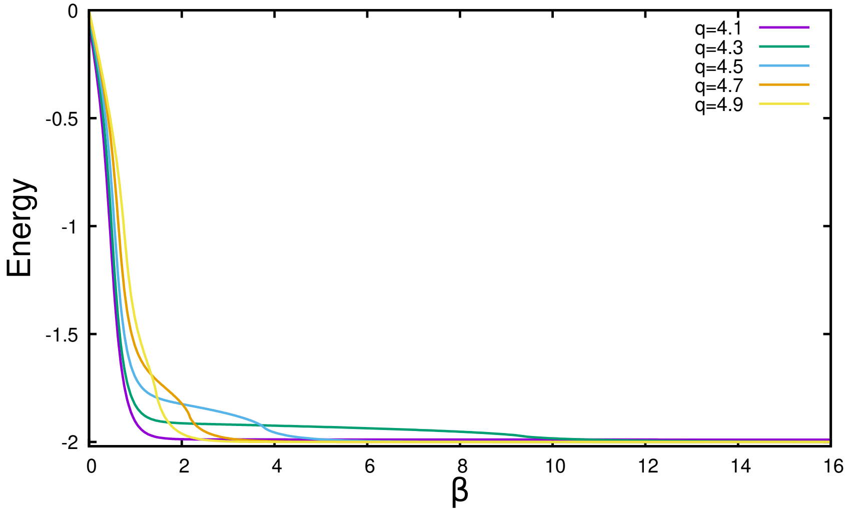
Validating TRG: $q = 4.5$



Validating TRG: $q = 4.9$

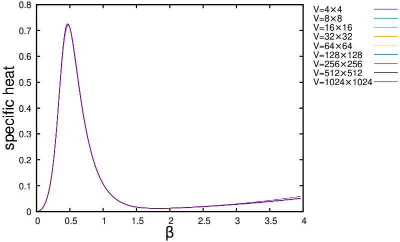


TRG results at large volume and large β

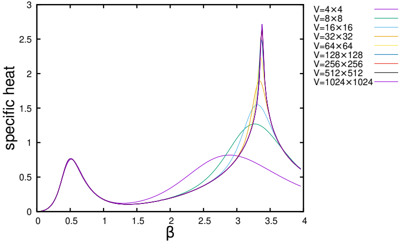


TRG results at large volume and large β

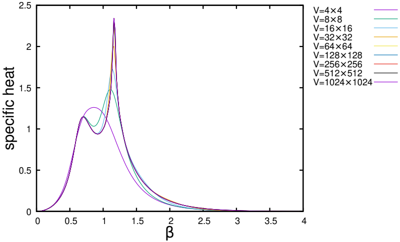
$q = 3.2$



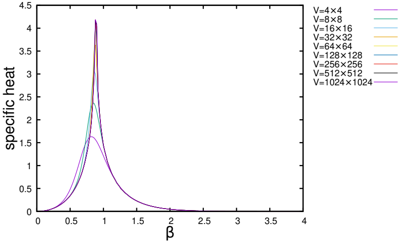
$q = 3.4$



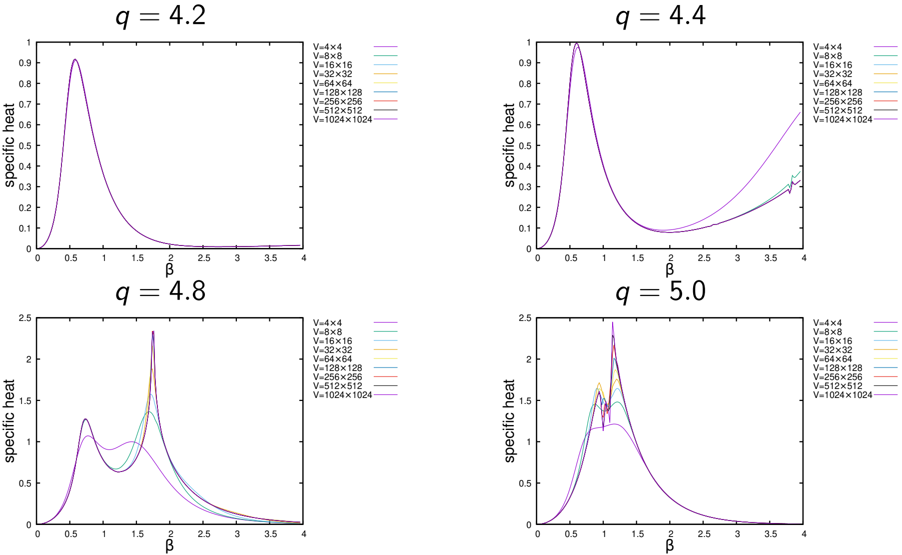
$q = 3.8$



$q = 4.0$

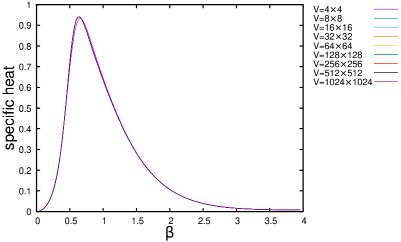


TRG results at large volume and large β

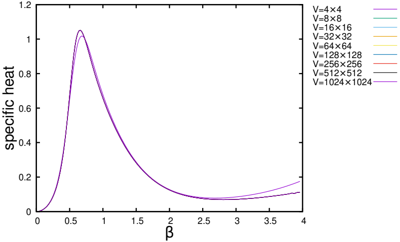


TRG results at large volume and large β

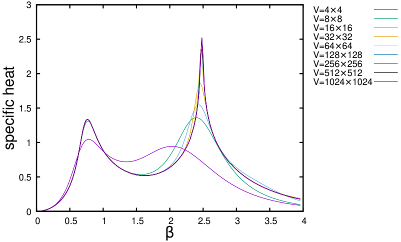
$q = 5.2$



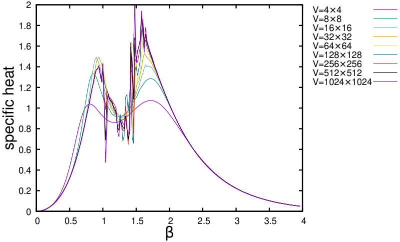
$q = 5.4$



$q = 5.8$



$q = 6.0$



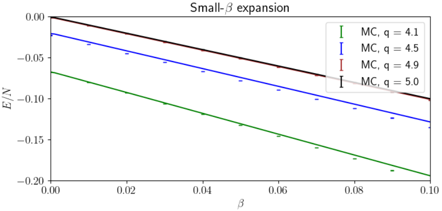
Small-β limit

- For non-integer q, the energy is offset from zero at $\beta = 0$, due to the broken $\mathbb{Z}$ symmetry but we understand this
- We can calculate exactly $E(\beta = 0)$
- Going further, Jin Zhang has calculated the small-β expansion:

$$Z = I_0^{2N}(\beta)\left[1 + 4NC_1^2 t_1(\beta) + \cdots\right]$$

where

$$C_k = \frac{1}{\lceil q \rceil}\mathrm{Re}\left(\frac{1 - e^{2\pi i k \lceil q \rceil / q}}{1 - e^{2\pi i k / q}}\right), \qquad t_n(\beta) = \frac{I_n(\beta)}{I_0(\beta)}$$



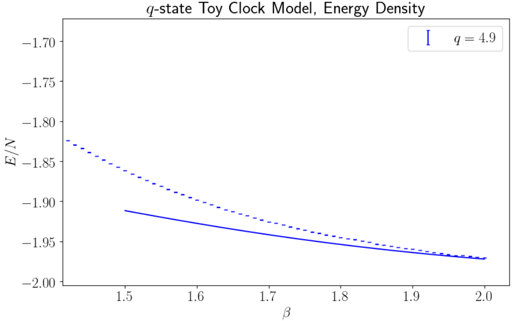
Large-β limit

For $q = 4.9$, partition function in order of increasing energy is

$$Z \simeq \frac{1}{\lceil q \rceil} e^{2N\beta} \left[1 + \frac{2N}{\lceil q \rceil} e^{-4\epsilon\beta} + \frac{8N}{\lceil q \rceil} e^{-4\tilde{\epsilon}\beta} + \frac{8N}{\lceil q \rceil} e^{-6\epsilon\beta} + \frac{32N}{\lceil q \rceil} e^{-6\tilde{\epsilon}\beta} + \cdots \right],$$

where

$$\epsilon = 1 - \cos\left(2\pi - 2\pi \cdot \frac{4}{4.9}\right), \qquad \tilde{\epsilon} = 1 - \cos\left(\frac{2\pi}{4.9}\right).$$



Quantum Hamiltonian

- Furthermore, Jin has developed a quantum Hamiltonian of this model, and has identified the second critical point (i.e. the Ising-like peak in this model)

$$\beta_c = \frac{\ln(1+\sqrt{2})}{1-\cos\psi}, \qquad \text{where} \qquad \psi = \frac{2\pi(\lceil q \rceil - 1)}{q}$$

- Works well when the small angle is sufficiently small
- So we come full circle in a sense
- With Jin's machinery, we hope to be able to put this model on a quantum simulator and study one of its interesting features

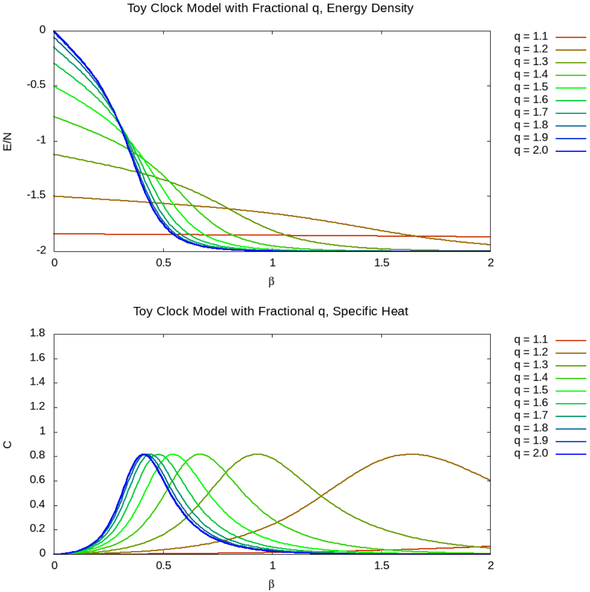
Summary

- Putting a clock model on a quantum simulator could be a good step in helping us understand how to eventually put quantum fields on a simulator
- This model with fractional q has some interesting features that might benefit from exploration on such a simulator
- However, we still have a little work to do
  - We want to study the magnetization and susceptibility and do proper finite size scaling
  - We want to enumerate all the states for a very small lattice and understand the exact cause of some of the features

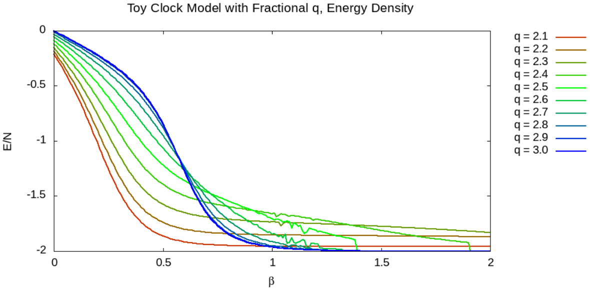
THE END

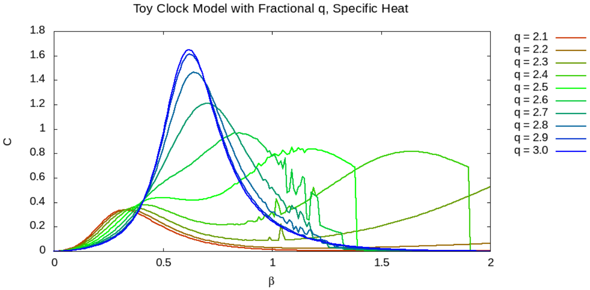
Overview

Toy Model: $1.1 \leq q \leq 2.0$

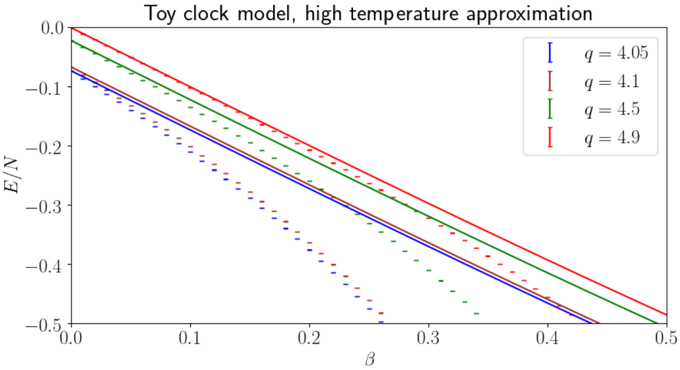


Toy Model: $2.1 \leq q \leq 3.0$





Toy Model: High temperature expansion



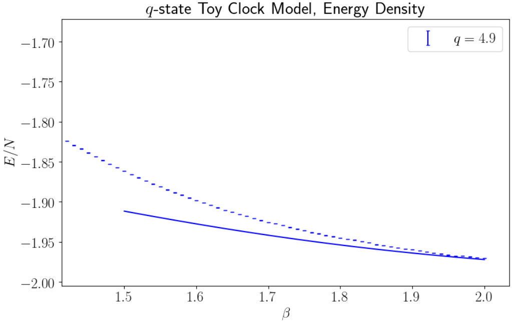
Toy Model: Low temperature expansion

For $q = 4.9$, partition function in order of increasing energy is

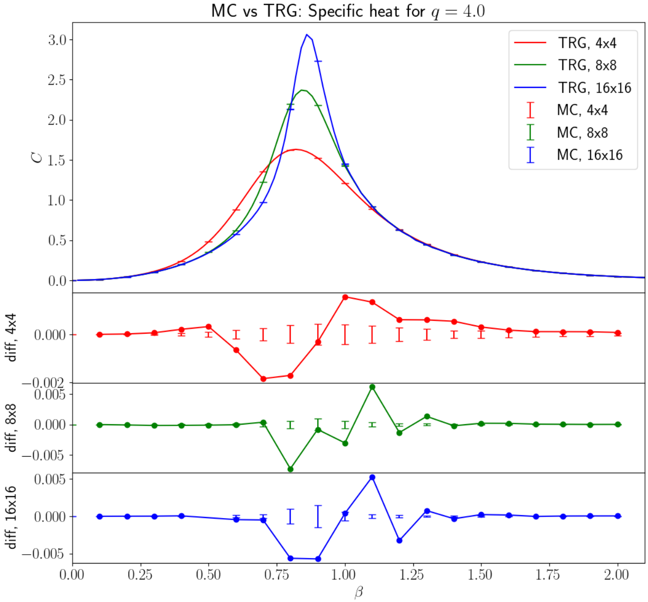
$$Z \simeq \frac{1}{\lceil q \rceil} e^{2N\beta} \left[1 + \frac{2N}{\lceil q \rceil} e^{-4\epsilon\beta} + \frac{8N}{\lceil q \rceil} e^{-4\tilde{\epsilon}\beta} + \frac{8N}{\lceil q \rceil} e^{-6\epsilon\beta} + \frac{32N}{\lceil q \rceil} e^{-6\tilde{\epsilon}\beta} + \cdots \right],$$

where

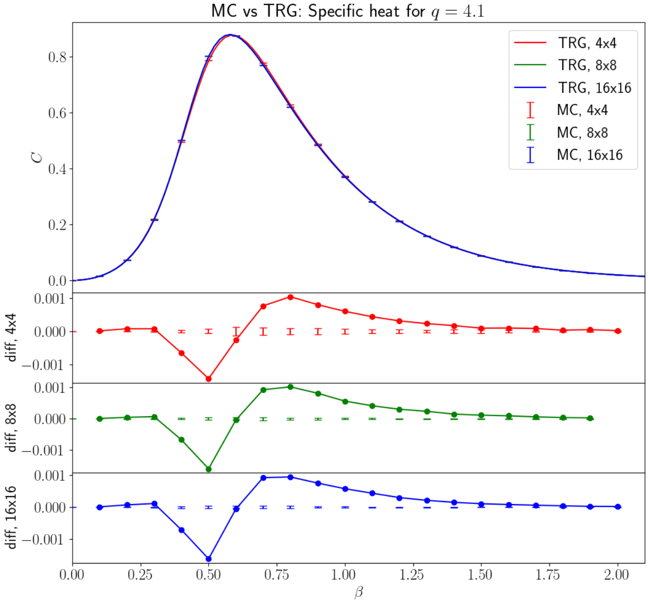
$$\epsilon = 1 - \cos\left(2\pi - 2\pi \cdot \frac{4}{4.9}\right), \qquad \tilde{\epsilon} = 1 - \cos\left(\frac{2\pi}{4.9}\right).$$



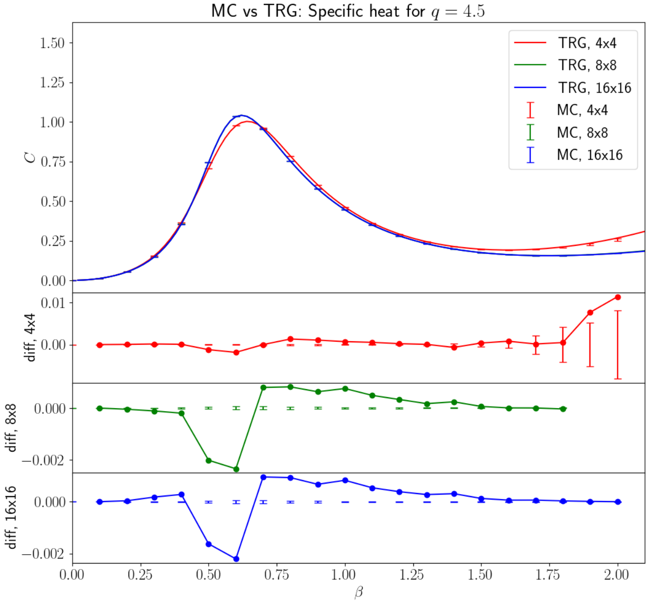
Validating TRG: $q = 4.0$



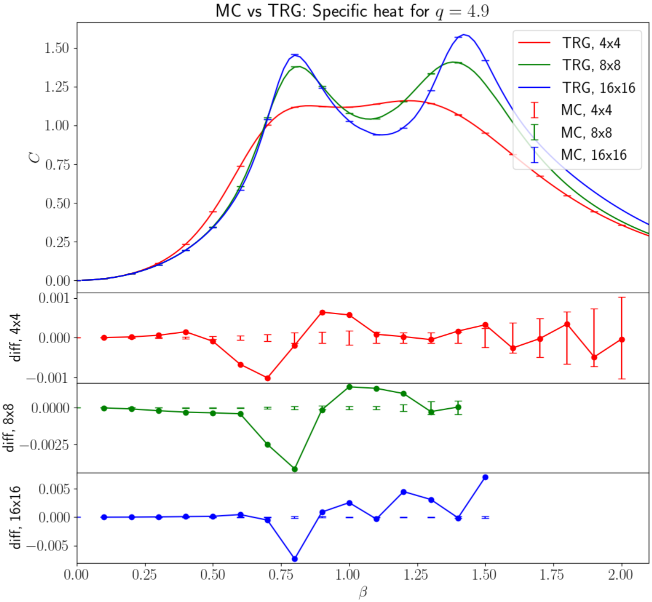
Validating TRG: $q = 4.1$



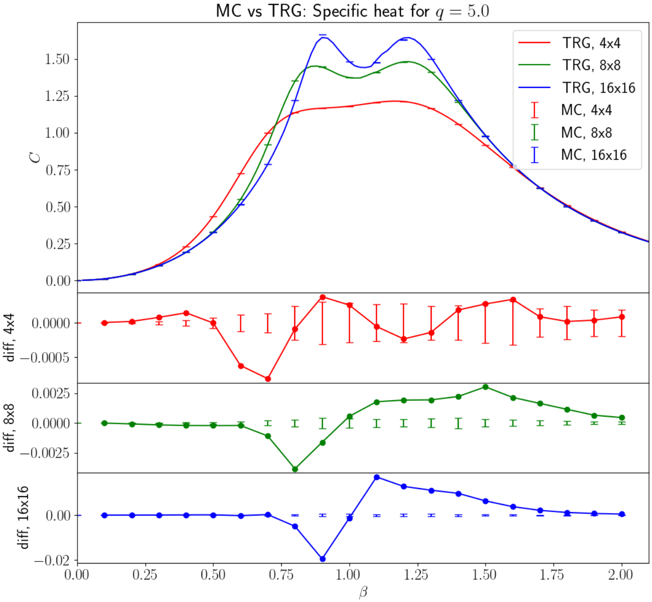
Validating TRG: $q = 4.5$



Validating TRG: $q = 4.9$



Validating TRG: $q = 5.0$



Overview

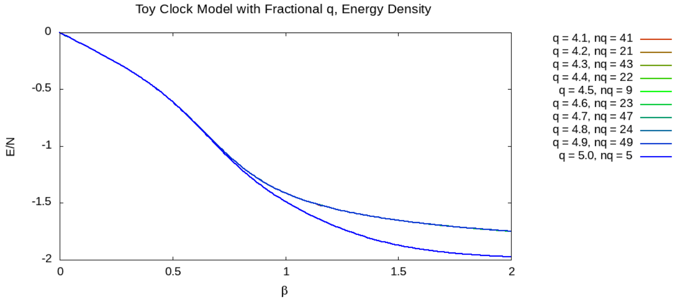
Toy Model with Large k

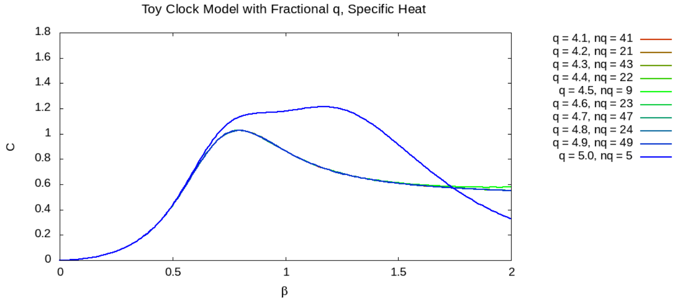
- We return to the energy function of the q-state clock model

$$E = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x),$$

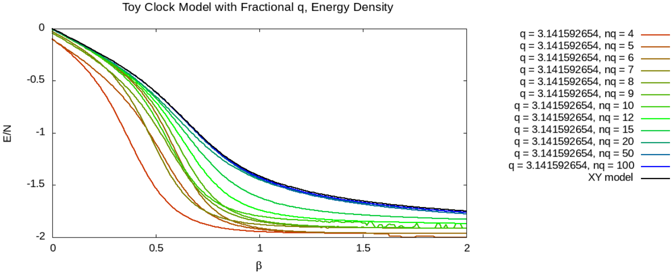
  where $\varphi_x \in 2\pi k/q$ with $q \in \mathbb{Z}$ and $k = 0, 1, \ldots, q-1$
- But now, we let $q \in \mathbb{R}$ and $k = 0, 1, \ldots$
- Note: k is no longer bounded above by $\lceil q \rceil$. Now we can let k go sufficiently large to recover periodicity

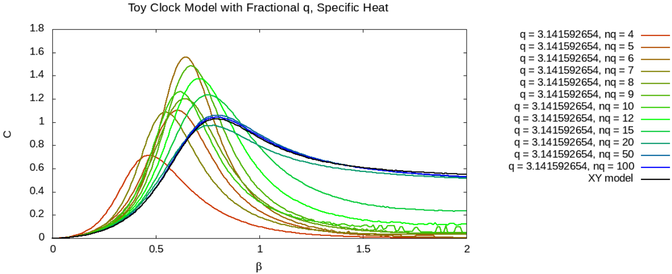
Toy Model with Large k





Toy Model with Large k





Overview

The Model

- XY model has energy function

$$E_{XY} = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x), \tag{1}$$

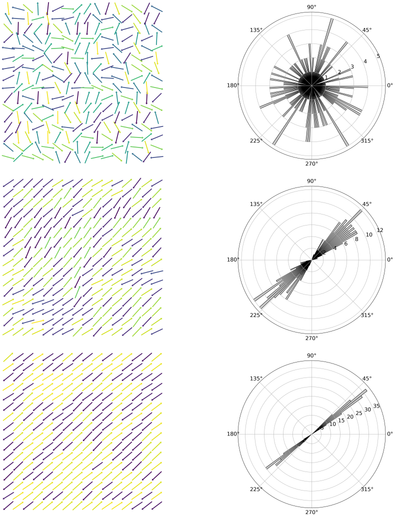
  where $\varphi_x \in [-\pi, \pi)$ and typically $J = 1$
- As proposed by Judah, we add a new term

$$E = -J \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_{x,\mu} \cos\left(q\left[\varphi_{x+\hat{\mu}} - \varphi_x\right]\right), \tag{2}$$

- When $\gamma \to \infty$, this model becomes the q-state clock model
- Here, one should be able to simulate also fractional q
- I have modified Berg's XY Metropolis code to evolve the system using (2)
- To get the properly normalized energy, I measure the energy using (1)

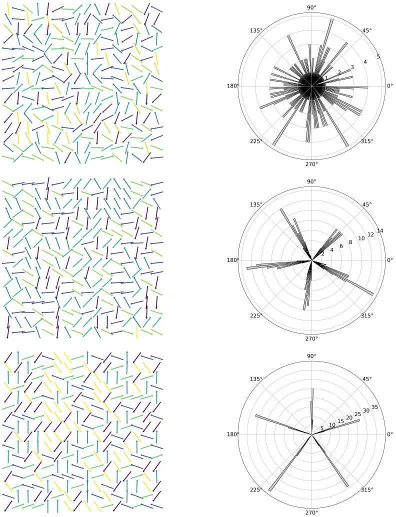
Checking Final States for $q = 2.0$

- Here, we run simulations (using random starts) on 16x16 lattices at $q = 2.0$ and $\beta = 0.1$ (i.e. in the disordered phase) with various γ and look at the final state after 2^{20} Metropolis sweeps
- In the top row, we have $\gamma = 0.0$, which reduces to the XY model
- In the middle row, we have $\gamma = 50.0$. We see the model beginning to favor 2 directions—approximating the 2-state clock model
- In the bottom row, we have $\gamma = 500.0$

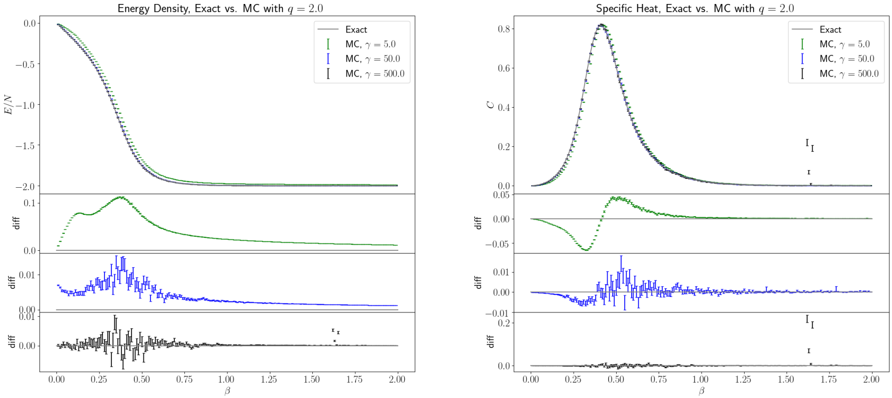


Checking Final States for $q = 5.0$

- Here, we run simulations (using random starts) on 16x16 lattices at $q = 5.0$ and $\beta = 0.1$ (i.e. in the disordered phase) with various γ and look at the final state after 2^{20} Metropolis sweeps
- In the top row, we have $\gamma = 0.0$, which reduces to the XY model
- In the middle row, we have $\gamma = 50.0$. We see the model beginning to favor 5 directions—approximating the 5-state clock model
- In the bottom row, we have $\gamma = 500.0$

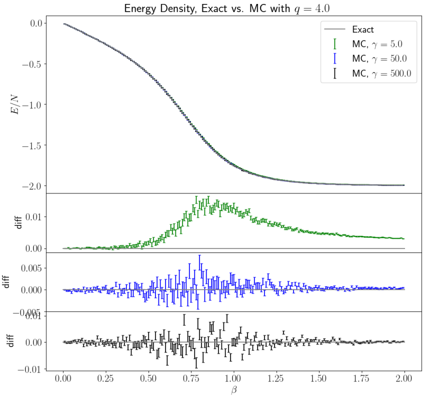


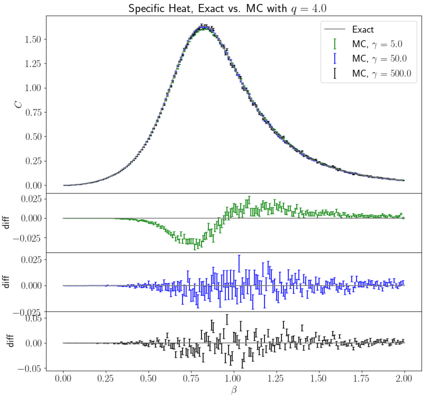
Energy Density and Specific Heat for $q = 2.0$



- We compare energy density (left) and specific heat (right) for the $q = 2.0$ model on 4x4 lattices at different γ with exact values
- Bottom plots show the differences from exact
- Anomalous results at large β for the $\gamma = 500.0$ case are due to insufficient equilibration

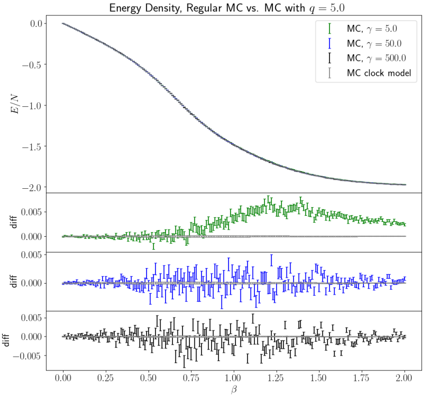
Energy Density and Specific Heat for $q = 4.0$

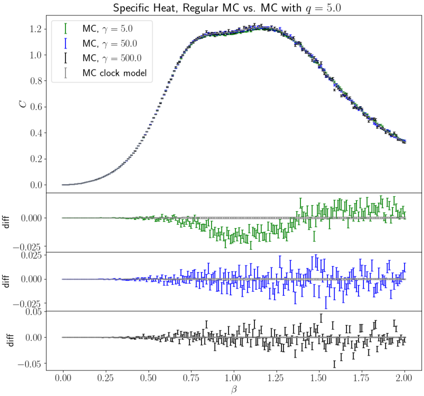




- We compare energy density (left) and specific heat (right) for the $q = 4.0$ model on 4x4 lattices at different γ with exact values
- Bottom plots show the differences from exact

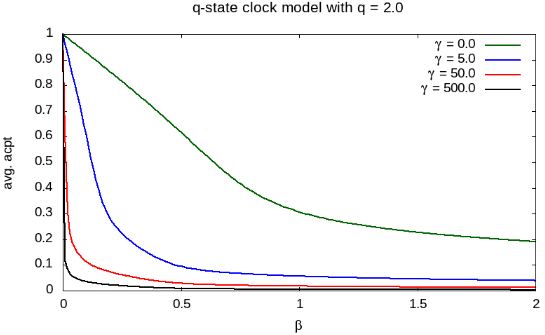
Energy Density and Specific Heat for $q = 5.0$





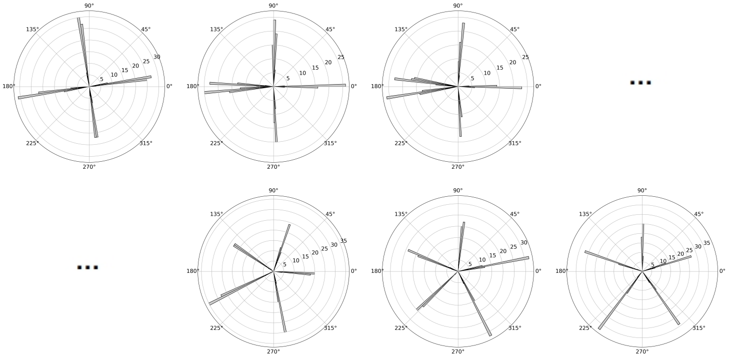
- We compare energy density (left) and specific heat (right) for the $q = 5.0$ model on 4x4 lattices at different γ with standard MC clock model
- Bottom plots show the differences from standard MC clock model since we don't have exact results for the 5-state clock model

Acceptance Rate



- Here, we show the dependence of the Metropolis acceptance rate on β and γ for the clock model with $q = 2.0$ on a 4x4 lattice
- As β is increased, acceptance drops significantly
- The effect is even worse as γ is increased.
- This is a problem (since autocorrelation becomes very large) that must be resolved before we can really look at lattices larger than 4x4

Fractional q: Angular distribution of final state spins



- Here we see how the angular distribution of the final lattice state evolves as q is increased
- We look at the final lattice state after 2^{20} Metropolis sweeps on 16x16 lattices with $\beta = 0.1$ and $\gamma = 500.0$
- The top row, from left to right, has $q = 4.0, 4.1, 4.2$
- The bottom row, from left to right, has $q = 4.8, 4.9, 5.0$

Fractional q: Energy Density and Specific Heat