

Pre-calculus
Class Notes

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Preface

About These Notes

My class notes can be found at www.leonhostetler.com/classnotes

Please bear in mind that these notes will contain errors. Any errors are certainly my own. If you find one, please email me at leonhostetler@gmail.com with the name of the class notes, the page on which the error is found, and the nature of the error.

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Chapter 1

Functions

A **function** is a special mathematical relationship that relates a unique output to every possible input. To see whether this condition is satisfied, use the **vertical line test**. If you graph the function and there is no place on the graph where a vertical line intersects the graph more than once, then it is a function. If any vertical line intersects the graph more than once, then it is not a function.

Even functions, such as $y = x^2$, are symmetrical about the y -axis. Odd functions, such as $y = x^3$, are symmetrical about the origin, which means the part in one quadrant can be “detached” and rotated around the origin and found to correspond exactly with the other part in the diagonal quadrant. To check algebraically whether a function is even or odd, replace all x 's with $-x$ and simplify. If you end up with the exact same function—it's even. If all signs have changed—it's an odd function. Otherwise, it's neither odd nor even. In other words

$$\begin{aligned}f(x) \text{ is even if } f(-x) &= f(x) \\f(x) \text{ is odd if } f(-x) &= -f(x).\end{aligned}$$

Polynomials in which all the terms have even exponents are even functions. Polynomials in which all the terms have odd exponents are odd functions. Polynomials with mixed exponents are neither odd nor even.

The **domain** of all polynomial function is all real numbers, \mathbb{R} . Perhaps the best way to understand domains is that a function will have all real numbers in its domain except those that would make the output impossible to compute (e.g. division by 0) or that would make the output impossible to display on a graph (e.g. imaginary numbers). This means that denominators of rational functions must be factored to see which values of x would result in division by zero and any functions containing radicals must be analyzed to see at what values of x the radical would result in imaginary numbers.

A **continuous** function is one that has no gaps.

A **one to one function** is one which every element of the range (y -values) corresponds to one and only one element of the domain (x values). A function is one to one if it passes both the vertical test and the horizontal test. If you graph the function and any horizontal line does not intersect the graph then the function is one to one. For example, the trig functions are not one to one unless we first restrict their domains to a single period.

An **inverse** function is a function that reverses another function. For example, $\pm\sqrt{x}$ is the inverse of x^2 because it undoes it. To find the inverse of a function, switch x 's and y 's and solve for y . Only one to one functions have inverses. If the function is not one to one, then you must restrict its domain so that it is one to one.

Tip

You can use your calculator to solve equations for specific values of x by putting your equation into **Y1** under the button **Y=**, and then using **VAR** > **Y-VARS** > **Y1(x)**. This also works with composed functions. If you put the outer function into **Y1** and the inner function into **Y2** then you can evaluate the composed function for a value of x using **VAR** > **Y-VARS** > **Y1(Y2(x))**.

Example 1.0.1

Find the inverse of

$$y = x^2 - 2.$$

Since this function is not one to one, we restrict the domain to either $(-\infty, 0]$ or $[0, \infty)$.

Switching x 's and y 's gives us $x = y^2 - 2$. Solving for y gives us the inverse function

$$y = \sqrt{x + 2},$$

if we use the restricted domain $[0, \infty)$, and

$$y = -\sqrt{x + 2},$$

if we use the restricted domain $(-\infty, 0]$.

The most important graphical property of inverse functions is that they are symmetric about the line $y = x$. When you invert a function, graph both of them to check that they are symmetric about $y = x$. Another useful fact is that the domain and range are switched in inverse functions.

If the function is denoted $f(x)$, then the inverse function is typically denoted $f^{-1}(x)$. For a function $f(x)$ and $g(x)$, the **composition** of the two is

$$(f \circ g)(x) = f(g(x)).$$

Example 1.0.2

Given $f(x) = x^2 - 2$ and $g(x) = 3x + 1$ find $(f \circ g)(2)$.

Composing the two functions using the definition given above gives us

$$(f \circ g)(x) = (3x + 1)^2 - 2 = 9x^2 + 6x - 1.$$

Evaluating the composed function at $x = 2$ gives us $(f \circ g)(2) = 47$.

A second way of doing this is to evaluate $g(2) = 7$ and plugging this result into $f(x)$ to get $f(7) = 47$.

Because of the definition of inverse functions, composing a function with its inverse gives x . For example, composing $f(x) = x^2$ with $g(x) = \sqrt{x}$ gives us $f(g(x)) = (\sqrt{x})^2 = x$. This means that you can check to see if a pair of functions are each other's inverse by composing them to see if the result is x .

The domain of a composed function $f(g(x))$ is the set of all numbers x that are in the domain of $g(x)$ such that $g(x)$ is in the domain of f . Think of it this way, $g(x)$ is found first, then $f(g(x))$ is found. Exclude the ones not in the domain of $g(x)$ and exclude the ones not in the domain of the composed function $f(g(x))$. The ones not in the domain of $f(x)$ are irrelevant when it comes to the domain of $f(g(x))$.

Example 1.0.3

Find the domain of $f(g(x))$ when $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x - 2}$.

The domain of $g(x) = \sqrt{x - 2}$ is $[2, \infty)$ so the domain of the composed function is bounded by this. Since all of these values are in the domain of $f(x) = \sqrt{x}$, the domain of the composed function $f(g(x))$ is just $[2, \infty)$.

Chapter 2

Exponential & Logarithmic Functions

Here are some properties of exponents that need to be remembered:

$$\begin{aligned}a^{-x} &= \frac{1}{a^x} = \left(\frac{1}{a}\right)^x \\a^{x+y} &= a^x a^y \\a^{x-y} &= \frac{a^x}{a^y} \\a^0 &= 1 \\a^x &= a^y \quad \text{iff } x = y.\end{aligned}$$

Exponential functions have the form

$$f(x) = Ca^x,$$

where a is the **growth factor** and C is the **initial value**.

If the ratios between consecutive outputs of a function are the same, then the function is exponential. In other words, if

$$\frac{f(x+1)}{f(x)} = a, \quad \text{or} \quad f(x+1) = af(x),$$

then $f(x)$ is an exponential function with base a .

Example 2.0.1

A population of bugs is tripling every 7 days. On the zeroth day, there was one bug. Find an exponential function that models the number of bugs versus the day.

The keyword is “tripling”, which implies that the base of our exponential function is $a = 3$. However, if x is the day, then $f(x) = 3^x$ isn’t right. Such a function would work for a population that is tripling every day, but not for one that is tripling every 7 days. To get the right values, we have to divide the exponent by 7

$$f(x) = 3^{\frac{x}{7}}.$$

Example 2.0.2

Now you have a different population of bugs. You know that on day 3, there are 2 bugs and the population doubles every 5 days. Find an exponential function that models the number of bugs versus the day.

Tip

When modeling with exponential equations, always start with a table with a column for x and a column for $f(x)$. Then from the information you're given in the problem, fill in the first couple rows of your table. That way, when you're building the function, you can check it against the values in your table to see if you're on the right track.

Since the population doubles periodically, $a = 2$. Since it doubles every 5 days, we have to divide the exponent by 5. We know that when $f(3) = 2$, $f(8) = 4$, $f(13) = 8$ and so on. Extrapolating backward, we realize that $f(-2) = 1$. In other words, the function has been shifted to the left by 2 units. Putting it all together gives us

$$f(x) = 2^{\frac{x+2}{5}}.$$

Example 2.0.3

In your final bug raising experiment, you start with 100 bugs on day 0. The population doubles twice a day (every 12 hours) so that at the end of day 1, the population is already 400. Find an exponential function that models the number of bugs versus the day.

Since it doubles, $a = 2$. Since it doubles twice in one day, the exponent must be multiplied by a factor of 2. Since you start with 100 bugs, $C = 100$. Putting it all together, we get

$$f(x) = 100 \cdot 2^{2x}.$$

Example 2.0.4

Solve

$$3^{3x+1} = 9^x.$$

If we put the 9 in the form 3^2 , then our equations becomes

$$3^{3x+1} = 3^{2x}.$$

One of the properties of exponential equations is that if the bases are the same then the exponents must be the same implying that $3x + 1 = 2x$, which we easily solve to get $x = -1$.

When dealing with an exponential function that has a fractional base, notice that

$$f(x) = \left(\frac{1}{a}\right)^x = a^{-x}.$$

The function $f(x) = a^{-x}$ looks the same as $f(x) = a^x$ except that it is reflected about the y -axis. Instead of starting low and growing exponentially, it starts high and decays exponentially.

The **domain** of exponential functions is all real numbers \mathbb{R} , and the **range** is all the numbers above the line $y = 0$.

To see the value that y approaches as x approaches infinity, replace x with ∞ and deduce the asymptote.

A logarithmic function is the inverse of an exponential function. In other words, it reverses what the exponential function does. To remember the relationship between exponential and logarithmic functions, you can remember

$$\begin{aligned} \text{base}^{\text{exponent}} &= \text{argument} \\ \log_{\text{base}} \text{argument} &= \text{exponent} \end{aligned}$$

Example 2.0.5

Put

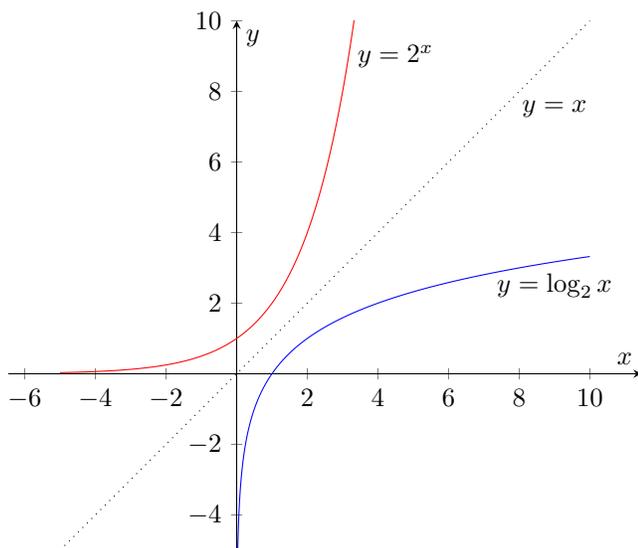
$$2^x = 8$$

into logarithmic form.

The answer is

$$\log_2 8 = x.$$

To graph the logarithmic function $y = \log_a x$, simply graph the associated function $y = a^x$ and then reflect it about the line $y = x$.



Here are some properties of logarithms that you should remember:

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$a^{\log_a M} = M$$

$$\log_a a^r = r$$

$$\log_a MN = \log_a M + \log_a N$$

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

$$\log_a M^r = r \log_a M$$

$$a^x = e^{x \ln a}$$

If $\log_a M = \log_a N$, then $M = N$. Vice versa is also true provided that M and N are positive real numbers. That is, if $M = N$ then $\log_a M = \log_a N$.

The **change of base formula** is

$$\log_a M = \frac{\log_b M}{\log_b a}.$$

Since your calculator only has buttons for logarithms of base e and base 10, to graph logarithms of other bases, you have to use the change of base formula.

With our current knowledge of algebra, some exponential equations can't be solved algebraically, but must be approximated using a calculator. For example, graph both sides of the equation and trace to the intersection point.

Because the logarithm is the inverse of the associated exponential function, the **domain** of a logarithm is the same as the range of the associated exponential equation. The key to finding the domain of a logarithm the key is to use the fact that the argument of a logarithm must be greater than zero. To find the domain, then, simply solve the inequality $arg > 0$.

Example 2.0.6

Find the domain of $y = \log_{10}(x - 3)$.

To find the domain, solve the inequality $x - 3 > 0$ to get $x > 3$, which tells us that the domain is

$$(3, \infty).$$

Example 2.0.7

Solve the logarithmic equation

$$\log_3(x) + \log_3(x - 6) = 3.$$

Using the properties of logarithms, we can combine the logarithms on the left to get $\log_3(x[x - 6])$, which gives us

$$\log_3(x^2 - 6x) = 3.$$

From the definition of a logarithm, we know that $3^3 = (x^2 - 6x)$, so

$$x^2 - 6x - 27 = 0,$$

which factors to

$$(x + 3)(x - 9) = 0,$$

so our solutions are $x = -3, 9$.

Remember that a logarithm cannot have a base of 1. Always check for extraneous solutions, that is, plug each of your solutions back into the original equation to ensure that they actually work.

Example 2.0.8

Solve the logarithmic equation

$$\log_2(x^2) = (\log_2 x)^2.$$

Simplifying the left side and expanding the right side gives us

$$2 \log_2(x) = (\log_2 x)(\log_2 x).$$

Dividing both sides by $\log_2(x)$ gives us

$$2 = \log_2(x),$$

which gives us the solution $x = 4$. However, because we divided both sides by $\log_2(x)$, we are neglecting the case where $\log_2(x) = 0$, which gives us $x = 1$. So our complete solution set is $x = 1, 4$.

Recall that **simple interest** is calculated as $I = prt$, where p is the principal, r is the interest rate, and t is the time.

Compound interest can be calculated by applying the simple interest formula repeatedly or by using the compound interest formula

$$A = P \left(1 + \frac{r}{n} \right)^{nt},$$

where A is the ending amount, P is the principal, r is the interest rate, t is the time in years, and n is the number of compoundings per year.

When interest is compounded continuously, it can be calculated using

$$A = Pe^{rt}.$$

Compound interest is when after a certain amount of time, the calculated interest is added back into the amount that is gathering interest. With simple interest, you don't earn interest on your interest earnings, but you do with compound interest. If you have an initial savings of P that is compounded yearly at an interest rate of r , then at the end of every year, your initial amount A increases by r , which is the same as multiplying it by $(1 + r)$. So if you invest P for t years, your total amount will be your initial amount multiplied by $(1 + r)$ an t number of times, that is, $A = P(1 + r)^t$.

If you compound more than once yearly, you use essentially the same formula, but you divide r by the number of times you're compounding in a year since interest r compounded 12 times a year is not the same as interest r compounded once a year for 12 years. In other words, r is the *annual* interest rate—not the rate for a month. So the formula becomes $A = P(1 + r/n)^{nt}$, where n is the number of compoundings per year, and t is the number of years.

If you're compounding continuously, you use the same formula, but you take n to infinity

$$A = \lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n} \right)^{nt} = Pe^{rt}.$$

The **effective rate of interest** is the simple interest rate that brings in the same amount of yearly interest as the compounded rate in question. In other words, 3% compounded monthly brings in the same interest in the first year as 3.042% simple interest. First solve for the interest earned using the compound formula then use the simple formula to get the equivalent simple rate.

$$r_e = \left(1 + \frac{r}{n} \right)^n - 1 = e^r - 1.$$

The effective interest rate allows you to easily compare different interest agreements.

Example 2.0.9

A person is getting 6% interest compounded weekly. What is the equivalent simple interest rate?

Plugging into the formula gives us

$$r_e = \left(1 + \frac{0.06}{52} \right)^{52} - 1 = 0.0617998,$$

so the person is getting the equivalent of a simple interest rate of 6.18%.

The present value of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in the specified time period. To figure it out, simply solve for P .

Tip

To get insight into a logarithmic equation, you can always graph both sides of the equation on your calculator then the solutions will be the x -values where the two graphs intersect. If the logarithms are in a base other than e or 10, you'll have to use the change of base formula when putting your functions into the calculator.

When an amount A of something varies over time according to the equation

$$A(t) = A_0 e^{kt},$$

it is said to follow the law of **uninhibited growth or decay**. In this equation, $A(t)$ is the amount at time t , A_0 is the initial amount at time $t = 0$, t is the time, and k is some constant that is given or has to be found. Many natural phenomena (and continuously compounded interest) follow this model. Radioactive materials follow the law of uninhibited decay.

Newton's law of cooling states that a heated object's temperature will decrease exponentially over time until it reaches the temperature of the surrounding medium. The equation that models this is

$$T(t) = T_s + (T_0 - T_s)e^{-kt},$$

where $T(t)$ is the temperature of the object at time t , T_0 is the initial temperature of the object at time $t = 0$, T_s is the constant temperature of the surroundings, and k is some constant.

In the real world, bacteria populations do not grow exponentially without bound. At some point, their growth will level off as they reach the limit of the petri dish containing them, or whatever. This kind of growth is called **logistic growth or decay** and the equation for it is

$$P(t) = \frac{cP_0}{P_0 + (c - P_0)e^{-rt}} = \frac{c}{1 + ae^{-rt}},$$

where $a = c/P_0 - 1$, c , and r are positive constants. In these equations, $P(t)$ is the population at time t , and $P_0 = c/(1 + a)$ is the initial population at time $t = 0$. Notice that it is also the y -intercept. Regular exponential functions have lower bounds (horizontal asymptote), but no upper bound. Logistic functions have both lower bounds and upper bounds. The upper bound is given by $P(t) = c$ and is called the **carrying capacity**. The constant r is the rate of growth when the population is small. The inflection point (where the graph changes "directions" and the rate of growth goes from increasing to decreasing) of a logistic growth model occurs at the point $((\ln a)/r, c/2)$.

Chapter 3

Quadratic Functions

Recall that a **quadratic function** has the form

$$f(x) = ax^2 + bx + c,$$

and that its solutions are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the leading coefficient of a quadratic is positive, then the function “opens” upward. If it is negative the function opens downward.

The zeros of a quadratic equation occur at

$$x = -\frac{b}{2a}.$$

To find its value, just plug it into the equation.

Example 3.0.1

Find the maximum or minimum of

$$f(x) = x^2 + 3x - 10,$$

and determine its value as well as whether it’s a maximum or a minimum.

Since $-b/(2a) = -3/2$ the maximum or minimum occurs at $x = -3/2$. Evaluating the function at that location gives us $f(-3/2) = -37/4$. Since the leading coefficient of the function is positive, the function opens upward, meaning this is a minimum. In summary, the minimum of $f(x)$ is $-37/4$ and occurs at $x = -3/2$.

Many **optimization** problems involve finding the vertex of a quadratic equation, since the vertex is located at the maximum or minimum of the quadratic. The x position of the vertex can also help you determine where a function is increasing and where it is decreasing, since it will be increasing on one side of the vertex and decreasing on the opposite side.

Example 3.0.2

Billy has 1000 feet of fencing. He will use his barn as one side of the fence. What is the maximum area that he can enclose.

We let w be the width of the fence, and l be the length that is parallel to his barn. This means $2w + l = 1000$. We want to maximize (i.e. optimize) the area, which is given by $A = lw$. Solving the first equation for l and substituting into the

second gives us the quadratic $A = w(1000 - 2w) = -2w^2 + 1000w$. Using $-b/(2a)$ to find the vertex gives us $w = 250$ plugging this into our first equation gives us $l = 500$, so the maximum area is

$$A = 250 \text{ ft} \cdot 500 \text{ ft} = 125\,000 \text{ sq ft}.$$

You can use substitution to solve a **quadratic-like equation** using the quadratic formula.

Example 3.0.3

Find the solutions of

$$x^4 - 7x^2 - 18 = 0.$$

Making the substitution $u = x^2$, we can rewrite this as the quadratic equation

$$u^2 - 7u - 18 = 0,$$

and when we use the quadratic formula, we get the solutions $u = -2, 9$. Now we must reverse our substitution. Since we said that $u = x^2$, we now have $x^2 = 9$ and $x^2 = -2$. The first one gives the solutions $x = -3, 3$. The second one has no real solutions.

Example 3.0.4

Find the solutions of

$$3(x - 2)^2 - 4(x - 2) + 16.$$

Making the substitution $u = x - 2$, we can rewrite this as the quadratic equation

$$3u^2 - 4u - 4 = 0,$$

and when we use the quadratic formula, we get the solutions $u = -2/3, 2$. Now we must reverse our substitution. Since we said that $u = x - 2$, we now have $x - 2 = -2/3$ and $x - 2 = 2$. The first one gives the solution $x = 4/3$, and the second one gives the solution $x = 4$.

The **marginal cost** of a product is the cost of producing one additional unit of the product.

Chapter 4

Polynomials

A **power function** is a function of the form

$$f(x) = Cx^r.$$

In this case, C is the coefficient of the power function and r is the degree. For example, $y = x$ and $y = 2x^3$ are power functions. The graphs of power functions always go through the point $(0, 0)$ and through two of the four points of the two by two square around the origin consisting of the points $(-1, 1)$, $(1, 1)$, $(1, -1)$, and $(-1, -1)$.

A **polynomial** is simply a sum of power functions. The **degree** of a polynomial is its largest exponent. For example, $f(x) = 2x^3 - 4x^2 + 1$ is a polynomial of degree 3.

If the leading coefficient of a polynomial is positive, then the function “opens” upward. If it is negative the function opens downward.

The **real zeros** of a polynomial correspond to the x -intercepts where the graph of the polynomial crosses or touches the x -axis. For example, the polynomial $f(x) = x^2 - x - 6$ factors to $f(x) = (x + 2)(x - 3)$ showing that it has two real zeros $x = -2$ and $x = 3$. These are the x -intercepts where the graph of the function crosses the x -axis.

The **multiplicity** of a zero is the number of times that it appears in the function. For example, $f(x) = x(x - 1)^2$ has a zero at $x = 0$ of multiplicity 1 and a zero at $x = 1$ of multiplicity 2. The multiplicity of a zero, can tell us how the graph behaves at that zero. If the multiplicity of a zero is odd, the graph will cross the x -axis there. If the multiplicity is even, the graph will only touch the x -axis there.

To figure out how a function behaves very near a specific zero, do nothing to that factor, but replace all other x 's with value of the zero. For example, with the function $f(x) = x^2(x - 3)$, if we want to determine how it acts very near the zero $x = 3$, rewrite it as $f(x) = 3^2(x - 3)$. This tells us that very close to $x = 3$ the function behaves like $f(x) = 3^2(x - 3) = 9x - 27$, or a line going through $x = 3$ with a slope of 9.

A **turning point** is a point where the graph of a function changes direction. In other words, a local maximum or minimum occurs at a turning point. If a polynomial has degree n , then the graph of the polynomial has at most $n - 1$ turning points. For example, a graph of degree 3 has at most 2 turning points. The converse is also true. If the graph of a polynomial has $n - 1$ turning points, the degree of the polynomial is at least n . For example, if you see a graph of a polynomial with 5 turning points, then the polynomial must be at least degree 6.

The **end behavior** of a graph is how it behaves to the far (infinite) left and the far (infinite) right. The end behavior of a polynomial resembles that of the graph of its leading term (i.e. the power function).

Tip

A graphing calculator has a **ZERO** function under the **CALC** menu that you can use to find the zeros of polynomials.

Example 4.0.1

Describe the end behavior of $f(x) = 3x^3 - x^2 + 5x - 3$.

On the left end, the graph will drop toward negative infinity and at the right end, the graph will rise toward positive infinity resembling the graph of $f(x) = 3x^3$ at both ends.

To find the intervals on which a polynomial is positive and negative, factor the polynomial to find its real zeros. Indicate these zeros with dots on a number line. These dots divide the number line into intervals. Test the polynomial with a value from each interval to see if the polynomial is positive or negative in that interval.

To graph a polynomial

1. Factor it to identify the zeros and multiplicities. Put dots on the x -axis of the Cartesian plane indicating the zeros.
2. Test each interval to see what the sign is of the function in that interval. Then update the Cartesian plane with this information.
3. Find the end behavior of the function using the leading term, and add arrows in the two quadrants of your graph indicating the end behavior.
4. Connect all the dots on your graph following the indicators.

Example 4.0.2

Graph the function

$$f(x) = x^3 - 4x^2 + 4x$$

using what you know about real zeros and their multiplicities as well as turning points and end behavior.

Since the largest term is x^3 , we know that the end behavior of the graph resembles that of $y = x^3$. Factoring the function gives us

$$f(x) = x(x - 2)^2,$$

so we know that it has the zeros $x = 0$ and $x = 2$. The $x = 0$ zero is of multiplicity 1, so it will cross the x -axis there. Since $f(x) = x(0 - 2)^2 = 4x$, we know that very close to $x = 0$, the graph will have a slope of 4. The $x = 2$ zero has a multiplicity of 2, so we know that it will only touch the x -axis there. Since $f(x) = 2(x - 2)^2$, we know that very close to $x = 2$, the graph will look like the graph of $y = 2(x - 2)^2$. With all this information, we can draw a fairly accurate graph of the polynomial.

Long division of polynomials is similar to regular long division.

Example 4.0.3

Divide $x^2 + 3x - 4$ by $x - 2$ using long division.

Using long division gives us

$$\begin{array}{r} x + 5 \\ x - 2 \overline{) x^2 + 3x - 4} \\ \underline{-x^2 + 2x} \\ 5x - 4 \\ \underline{-5x + 10} \\ 6 \end{array}$$

so

$$\frac{x^2 + 3x - 4}{x - 2} = x + 5 + \frac{6}{x - 2}.$$

When dividing polynomials, remember the terms

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}.$$

The **division algorithm** states that

$$P(x) = (x - c)Q(x) + R,$$

states that for a polynomial $P(x)$ of at least degree 1 and any number c , there is another polynomial $Q(x)$ (i.e. the quotient), with a degree one less than the degree of $P(x)$, such that $x - c$ times $Q(x)$ plus a number R , gives the polynomial $P(x)$. Another way of putting this is

$$\frac{P(x)}{x - c} = Q(x) + \frac{R}{x - c}.$$

The **remainder theorem** states that if a polynomial $P(x)$ is divided by $x - c$, then the remainder is $P(c)$

$$\frac{P(x)}{x - c} = Q(x) + \frac{P(c)}{x - c}.$$

To find the remainder when dividing a polynomial by a divisor of the form $x - c$, it is much faster and easier to use this theorem and simply find $P(c)$ than it is to use long or even synthetic division.

The **factor theorem** is a consequence of the remainder theorem and states that

- If $f(c) = 0$, then $x - c$ is a factor of $f(x)$
- If $x - c$ is a factor of $f(x)$, then $f(c) = 0$

The **fundamental theorem of algebra** states that a polynomial of degree n has n complex zeros. In other words, a polynomial of degree n can have up to but no more than n real zeros. Each multiplicity of a real zero is counted as a zero itself.

The **conjugate pairs theorem** states that if a function has all real coefficients and it has a complex zero $z = a + bi$, then the complex conjugate $z^* = a - bi$ is also a zero of the function. This makes sense when you realize that if the complex zeros didn't come in pairs then the function would have a complex coefficient instead of all real coefficients. A corollary of this theorem is that odd functions with real coefficients must have at least one real zero because it can't have an odd number of complex zeros.

According to the **rational zeros theorem**, for a polynomial of degree 1 or higher with integer coefficients, all the real zeros will be of the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient. In other words, all the real zeros will be a subset of the possibilities

$$\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}.$$

Keep in mind that this theorem only gives the possible rational zeros of a polynomial. It is possible (example $f(x) = x^2 - 2$) that the polynomial has no rational roots—they could be irrational.

Example 4.0.4

Using the rational zeros theorem, what are all the possible real zeros of

$$f(x) = 2x^3 + 4x^2 + 2x - 8?$$

The factors of the constant term -8 are $\pm 1, \pm 2, \pm 4,$ and ± 8 . The factors of

Tip

When doing synthetic division, you have to list *all* of the ordered coefficients in the first line even if some are zero and thus are typically not written in the expression.

the leading coefficient 2, are ± 1 and ± 2 , so the possible real zeros are

$$\pm \frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8.$$

By testing these possibilities using long division,

$$\begin{array}{r} 2x^2 + 6x + 8 \\ x - 1 \overline{) 2x^3 + 4x^2 + 2x - 8} \\ \underline{- 2x^3 + 2x^2} \\ 6x^2 + 2x \\ \underline{- 6x^2 + 6x} \\ 8x - 8 \\ \underline{- 8x + 8} \\ 0 \end{array}$$

we find that $x = 1$ is a zero of the polynomial. By applying the quadratic formula to the quotient $2x^2 + 6x + 8$, we see that there are no more real zeros—the remaining two zeros must be complex.

To perform **synthetic division** of a polynomial $f(x)$ by $x - a$, we

1. start by listing the ordered coefficients of $f(x)$ in a horizontal line, then
2. drawing a vertical separating line to the left and a horizontal separating line two spaces below.
3. Then we write a on the left side of the vertical line.
4. To begin the process of synthetic division, we copy the first coefficient below the horizontal separating line. This is the first ordered coefficient of the quotient, and it will have a degree that is one less than the degree of $f(x)$.
5. To get the next ordered coefficient, we multiply the first one by a , write the result in the second row and second column of the box. Then our second coefficient is the sum in the second column.

The best way to remember it is to work through an example or two.

Example 4.0.5

Calculate

$$\frac{2x^3 + 4x^2 + 2x - 8}{x - 1},$$

using synthetic division.

Synthetic division gives us

$$1 \left| \begin{array}{cccc} 2 & 4 & 2 & -8 \\ & 2 & 6 & 8 \\ \hline 2 & 6 & 8 & 0 \end{array} \right.,$$

which tells us that

$$\frac{2x^3 + 4x^2 + 2x - 8}{x - 1} = 2x^2 + 6x + 8.$$

The remainder is zero, which implies that $x - 1$ is a factor of $2x^3 + 4x^2 + 2x - 8$, and $x = 1$ is a zero of $f(x) = 2x^3 + 4x^2 + 2x - 8$.

Example 4.0.6

Calculate

$$\frac{2x^3 + 4x^2 + 2x - 8}{x + 1},$$

using synthetic division.

Synthetic division gives us

$$-1 \left| \begin{array}{cccc} 2 & 4 & 2 & -8 \\ & -2 & -2 & 0 \\ \hline 2 & 2 & 0 & -8 \end{array} \right.,$$

which tells us that

$$\frac{2x^3 + 4x^2 + 2x - 8}{x + 1} = 2x^2 + 2x + \frac{-8}{x + 1}.$$

The remainder is not zero, which implies that $x + 1$ is not a factor of $2x^3 + 4x^2 + 2x - 8$, and $x = -1$ is not a zero of $f(x) = 2x^3 + 4x^2 + 2x - 8$.

Tip

Remember when dividing a polynomial by $x - a$, you use a , not $-a$ in the synthetic division.

Using the rational zeros theorem in conjunction with the remainder theorem gives us a powerful tool for factoring polynomials. We can use the rational zeros theorem to find the potential zeros and then use the remainder theorem to test each of those potential zeros to see if they are actual zeros. The degree of the polynomial tells you the maximum number of zeros that it might have. One thing to keep in mind is that if any of the zeros has a multiplicity greater than one, this method won't tell you that. If you also graph the function you can tell approximately where the zeros are and you can deduce their multiplicities.

Example 4.0.7

Find all the real zeros of

$$f(x) = x^4 + 2x^3 - 16x^2 - 2x + 15.$$

From the rational zeros theorem, we know that the potential zeros are

$$\pm 1, \pm 3, \pm 5, \pm 15.$$

From the degree of the polynomial, we know that there will be a maximum of four zeros. We start testing the potential zeros using the remainder theorem. If there is no remainder, then we know we have found a zero.

$$\begin{aligned} f(1) &= 0 \\ f(-1) &= 0 \\ f(3) &= 0 \\ f(-3) &= 96 \\ f(5) &= 480 \\ f(-5) &= 0. \end{aligned}$$

So our zeros are $x = -5, -1, 1, 3$. We can stop testing because we've already found four zeros, and we know there can't be more than that.

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible quadratic factors. In other words, no matter the degree of the polynomial it can be factored such that none of the factors are higher than degree two.

Theorem: A polynomial function of odd degree with real coefficients, has at least one real zero. Think of it graphically. Since any odd function crosses the x -axis, then it must have a real zero.

The **bounds on real zeros theorem** allows us to find an upper and lower bound on the real zeros of a polynomial. This helps us when it comes to looking for those zeros. It also tells us the maximum values for our calculator viewing screen to ensure that were not missing any of the zeros. To use this theorem

1. If the polynomial does not have a leading coefficient of one, convert it to one that does by dividing every term by the leading coefficient.
2. Write down all the coefficients (including the constant), discard the signs, and throw away the leading coefficient.
3. Select the largest value and add 1 to it. This is *bound 1*.
4. Choose which is larger, the value 1 or the sum of all the values from step 2. This is *bound 2*.
5. The smallest of bound 1 and bound 2 is what you're looking for. All the real zeros of the polynomial will occur between plus and minus that number.

Example 4.0.8

Using the bounds on real zeros theorem, find a number n such that all the real zeros of

$$f(x) = 6x^5 + 4x^3 - x^2 - 7$$

occur within $x = \pm n$.

Since the leading coefficient is not 1, we divide each term by the leading coefficient to get a list of coefficients $\{1, 4/6, -1/6, -7/6\}$. Discarding the signs and throwing away the leading coefficient gives us $\{4/6, 1/6, 7/6\}$. Selecting the largest value and adding one to it gives us *bound 1* = $13/6$. Since $4/6 + 1/6 + 7/6 = 2 > 1$, our value for *bound 2* is 2. Since *bound 2* < *bound 1*, the BOUND we are looking for is 2, and so

$$n = 2,$$

and all the real zeros of the given polynomial must lie in the interval $-2 < x < 2$.

The **intermediate value theorem** states that if there is a sign change between $f(a)$ and $f(b)$ for a polynomial then there is a real zero between $x = a$ and $x = b$. This is of course obvious since a polynomial is continuous so if it's positive at one part and negative at another part then it must cross the x -axis at some place between those two parts.

If a polynomial has an irrational zero, its value can be approximated using the intermediate value theorem. Just make the interval $[a, b]$ smaller and smaller while ensuring that $f(a)$ has opposite sign from $f(b)$.

4.1 Factors, Zeros, and Solutions

The following statements about a polynomial $f(x)$ are all equivalent statements. That is, if any of them is true, they must all be true.

- $x = a$ is a **zero** of $f(x)$.
- $(x - a)$ is a **factor** of $f(x)$.
- $x = a$ is a **solution** of the equation $f(x) = 0$.

Consider the function $f(x) = 2x^3 + 4x^2 + 2x - 8$. This function has a zero at $x = 1$, which means its graph crosses the x -axis at $x = 1$. Also, $f(x) = 2x^3 + 4x^2 + 2x - 8 = (x - 1)(2x^2 + 6x + 8)$, so $x - 1$ is a factor of $f(x)$. Finally, we have that $f(1) = 0$, which means $x = 1$ is a solution to the equation $f(x) = 0$.

Tip

The difference quotient is just the first derivative if h is made to approach zero.

Chapter 5

Rational Functions and Difference Quotients

A **tangent line** is a line that just touches the curve of a graph. That is, it intersects the graph at exactly one point. A **secant line** is a line that cuts through a portion of the graph. That is, it intersects the graph at at least two points.

The **difference quotient**

$$\frac{f(x+h) - f(x)}{h},$$

gives the slope of the secant line connecting the points $(x, f(x))$ and $(x+h, f(x+h))$.

The difference quotient gives the **average rate of change** between the two points $(x, f(x))$ and $(x+h, f(x+h))$ on a graph. For a linear function, this is just the slope of the line. For a non-linear function, this is the slope of the secant line connecting the two points.

Example 5.0.1

Using the difference quotient, find the slope of the secant line m_{sec} for any x for the function $10x^2 - 2x + 8$.

The difference quotient is

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h}.$$

Finding $f(x+h)$, we get

$$f(x+h) = 10(x+h)^2 - 2(x+h) + 8 = 10x^2 + 20hx + 10h^2 - 2x - 2h + 8.$$

Plugging it into the difference equation and simplifying gives us

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h} = \frac{20hx + 10h^2 - 2h}{h} = 20x - 2 + 10h.$$

A **rational function** is a function that is a ratio of polynomial functions. For example,

$$f(x) = \frac{x-2}{x^2+3x},$$

is a rational function.

A **proper** rational function is one in which the degree of the numerator is less than the degree of the denominator. Otherwise, it's an **improper** rational function.

For a rational function $f(x)$ in lowest terms

- The x -intercepts are the real zeros of the numerator

- The y -intercept is $f(0)$
- The vertical asymptotes are the real zeros of the denominator

Finding the vertical asymptotes is as easy as factoring the denominator, but you have to be careful. If the same factor exists in the numerator, the two will cancel out and there will be no horizontal asymptote there. For example,

$$f(x) = \frac{x^2 - 4}{x(x+2)} = \frac{(x+2)(x-2)}{x(x+2)},$$

has only one vertical asymptote—at $x = 0$.

Finding the **horizontal asymptote(s)** of a rational function $f(x) = P(x)/Q(x)$ can be a little more difficult.

1. If the degree of the numerator is less than the degree of the denominator (i.e. it is proper), then the horizontal asymptote is $y = 0$
2. If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is

$$y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}.$$

3. If the degree of the numerator is greater than the degree of the denominator then the horizontal asymptote is actually a slant asymptote and must be found by performing long division such that you end up with a polynomial plus a proper rational remainder of the form

$$\frac{P(x)}{Q(x)} = g(x) + \frac{r(x)}{q(x)}.$$

- a) If $g(x) = b$, that is, it's a constant, then $y = b$ is the horizontal asymptote. This only occurs if the degree of the numerator is the same as the degree of the denominator.
- b) If $g(x) = mx + b$, that is, it's a linear function, then $y = mx + b$ is the slant asymptote. Notice that the remainder $r(x)/q(x)$ is not part of the slant asymptote.
- c) In all other cases, there are no horizontal or slant asymptotes. This means if the degree of the numerator is more than 1 greater than the degree of the denominator, then there is no horizontal or slant asymptote.

Note: Always make sure to reduce rational functions to lowest terms before finding the asymptotes.

Example 5.0.2

Discuss the characteristics of

$$f(x) = \frac{3x^3 + 8x^2 - 7x - 11}{x^2 + 2x - 4}.$$

Plugging in $x = 0$ tells us that the y -intercept is $11/4$. Using the quadratic formula on the denominator tells us that the vertical asymptotes occur at $x = -1 \pm \sqrt{5}$. Since the degree of the numerator is greater than the degree of the denominator, we have to perform long division to find the slant asymptote. Doing so gives us

$$f(x) = 3x + 2 + \frac{x - 3}{x^2 + 2x - 4},$$

so there is a slant asymptote along the line $y = 3x + 2$. Finally, to complete the graph, test the original function at a place inside each interval to figure out how the graph is shaped in that interval.

To construct a rational polynomial from a given graph, remember the multiplicities of the zeros of the numerator define whether or not the graph touches $x = 0$ or crosses it. The multiplicities of the zeros of the denominator are also useful in figuring out the behavior of the graph as it approaches the asymptotes. If the parts of the graph that are approaching an asymptote from either side are symmetrical with respect to that asymptote, then the zero of the denominator that is causing that asymptote must have an even multiplicity (meaning that $f(x)$ doesn't change sign immediately after x crosses the asymptote).

Chapter 6

Vectors

A **scalar** is a regular number with only magnitude. A **vector** is a mathematical object that contains both a magnitude and a direction. For example, 30 miles straight north is a vector because it has a direction (straight north) and a magnitude (30 miles). On a coordinate system, a vector is represented as an arrow pointing in a specific direction. The length of the arrow is the magnitude of the vector and the direction of the vector can be specified by the angle between the arrow and the horizontal. The **magnitude** of a vector is simply the length of the vector.

To denote that a quantity is a vector, we use a large bold letter such as **V** or a large bold letter with an arrow on top \vec{V} . The magnitude of a vector is denoted by putting it between vertical bars $|\vec{V}|$.

The sum of vectors is called the **resultant vector**. To sum vectors, set the one vector's tail to the other vector's head. The resultant vector is the one from the tail of the first to the head of the second. To subtract vectors, set both tails together and the resultant vector is the one from head to head. Alternatively, you can add the two but reverse the direction of the one being subtracted.

A vector in **standard position** has its tail at the origin of the Cartesian coordinate system. Given a vector, in standard position, with a specified magnitude and degrees it is easy to use trig functions to break the vector into its horizontal and vertical vector components.

For a vector \vec{V} , its magnitude and its x and y components are given by

$$\begin{aligned}V_x &= |\vec{V}| \cos \theta \\V_y &= |\vec{V}| \sin \theta \\|\vec{V}| &= \sqrt{V_x^2 + V_y^2}\end{aligned}$$

where θ is the angle between the vector in standard position and the positive x -axis.

You can use the calculator to find vector components, and the resultant vector given its components. Use the features under **2nd > Angle**.

- **P ► Rx**, input (magnitude, angle) to receive the x -component (i.e. horizontal component of the vector)
- **P ► Ry**, input (magnitude, angle) to receive the y -component (i.e. vertical component of the vector)
- **R ► Pr**, input (x -component, y -component) to receive the magnitude of the resultant vector
- **R ► P θ** , input (x -component, y -component) to receive the angle of the resultant

In physics, a **force** is often represented by a vector since forces have both magnitude and direction.

When an object is not moving, it is said to be in **static equilibrium**. When an object is in static equilibrium, it means that all the forces on it cancel, so that there is no net force. If there was a net force on the object, then it would be moving. To solve static equilibrium problems treat each force on the object as a separate vector. Then choose a convenient coordinate system and break each vector into x -components and y -components. Then since the object isn't moving, all the x components sum to zero and all the y -components sum to zero, so you can solve for this missing information by using a system of equations.

When it comes to static equilibrium problems remember that **weight** is a force—it is mass times the gravitational acceleration.

Work is a physics term defined as the force \vec{F} applied to an object times the distance d that the force causes the object to move

$$W = |\vec{F}|d \cos \theta,$$

where θ is the angle between the force and the object's resulting direction of motion.

Note: The physics definition of work has very little in common with the common definition of work. For example, there is work involved in moving an extremely light object a very tiny distance, but there is no work involved in holding an extremely heavy object at some height for a time.

A vector in standard position with the endpoint of (a, b) can be represented as

$$\vec{V} = \langle a, b \rangle.$$

This is called the **component form** because a is the horizontal component and b is the vertical component. The magnitude of \vec{V} can easily be found using the Pythagorean theorem on a and b .

Vectors of this form can be added and subtracted simply by adding/subtracting their components.

Vectors of this form can be multiplied by a scalar simply by multiplying each of the components by the scalar.

A unit vector is any vector with a magnitude of 1. The unit vector along the x -axis is called the unit horizontal vector and is denoted by \hat{i} and the unit vector along the y -axis is called the unit vertical vector and is denoted by \hat{j} .

Any vector can be written as the sum of the above unit vectors. For example

$$\vec{V} = \langle 1, 2 \rangle = \hat{i} + 2\hat{j}.$$

Writing vectors in this form is called **vector component form**. Vectors of this form can be added, subtracted, and multiplied by scalar multiples in the same way as vectors in component form.

The **dot product** or **scalar product** is a special product of two vectors. Given the vectors $\vec{A} = \langle a_x, a_y \rangle$ and $\vec{B} = \langle b_x, b_y \rangle$ that have an angle θ between them, their dot product is

$$\begin{aligned}\vec{A} \bullet \vec{B} &= a_x b_x + a_y b_y \\ \vec{A} \bullet \vec{B} &= AB \cos \theta.\end{aligned}$$

Notice that the dot product of two vectors is a scalar, that is, it's a regular number. The dot product is often used to find the angle between two vectors.

The dot product of a vector with a unit vector gives the **projection** of the vector along the direction of the unit vector.

Chapter 7

Matrices

A **matrix** is a mathematical object consisting of rows and columns of elements. An $m \times n$ matrix has m rows and n columns. If $m = n$, it's called a **square matrix**.

For two matrices to be equal, they must have the same number of rows, the same number of columns, and every entry in one matrix must be the same as the corresponding entry in the other.

To add two matrices **A** and **B**, they must have the same number of rows and columns. To add them, simply add each entry in **A** to the corresponding entry in **B** to create a new matrix **C**. Subtracting is just the same except you subtract the entries from each other. Obviously, with subtraction, it matters which matrix is subtracted from the other.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 9 \end{bmatrix}.$$

A **zero matrix** is one in which all the elements are zero.

To multiply a matrix by a scalar, simply multiply every element of the matrix by the scalar.

$$2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

The **determinant** of a 2×2 matrix is calculated as

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The determinant of a 3×3 matrix is calculated as

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Each two by two determinant above is known as a **minor**. To find a minor of an entry, simply build a determinant using every record except the ones in the same row or column as the given entry.

Notice that the determinant of a matrix is denoted by vertical lines whereas a matrix is denoted by brackets.

The value of a determinant changes sign if any two rows or columns are interchanged.

If the entries of any row or column are multiplied by a factor k , then the resulting determinant will also be multiplied by a factor of k .

If any row or column is multiplied by a factor k and the results are added to the corresponding entries in another row or column, then the determinant is unchanged.

If the entries in an entire row or column are zeros, then the determinant is zero.

Determinants can also be done on a calculator. This is especially helpful in checking your work when finding the determinant of 3×3 matrices. Simply go to **2nd > Matrix > Edit** to enter the details then go to **2nd > Matrix > Math > det(** and enter the matrix you filled by going into **2nd > Matrix**.

A **row vector** is a $1 \times n$ matrix and a **column vector** is an $n \times 1$ matrix. To multiply a row vector by a column vector, multiply the first entry in the row vector with the first entry in the column vector and add that to the product of the second entries, and so on. A row and a column vector can be multiplied only if they have the same number of entries.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \times \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}.$$

Notice that the product of a row vector by a column vector is just a pure number.

For two matrices to be multiplied with each other, the number of columns in **A** must equal the number of rows in **B**. If **A** is an $m \times r$ matrix and **B** is an $r \times n$ matrix, the product **AB** will be an $m \times n$ matrix. For **AB = C**, the entry in the i th row of **C** and the j th column is the product of the i th row of **A** times the j th column of **B**. Note, that order of multiplication matters. Matrix multiplication is not commutative. In fact, even if **AB** and **BA** are both defined, they will rarely be equal. The associative and distributive properties do hold for matrix multiplication.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} ag + bj & ah + bk & ai + bl \\ cg + dj & ch + dk & ci + dl \\ eg + fj & eh + fk & ei + fl \end{bmatrix}.$$

An **identity matrix** is a square matrix in which the main diagonal (from top left to bottom right) is nothing but ones and the rest of the elements are zeros. We denote the 2×2 identity matrix as I_2 and the 3×3 identity matrix as I_3 .

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For an $n \times n$ matrix **A**, $I_n \mathbf{A} = \mathbf{A} I_n = \mathbf{A}$. More generally, for an $m \times n$ matrix **A**, $I_m \mathbf{A} = \mathbf{A} I_n = \mathbf{A}$.

A square matrix multiplied by its **inverse matrix**, yields the identity matrix of the same size. In this special case multiplication is commutative—order doesn't matter. A matrix that has an inverse is called a nonsingular matrix and a matrix that has no inverse is called a **singular matrix**.

If the determinant of a matrix is zero then the matrix is singular.

An **augmented matrix** is a matrix created by appending the columns of two matrices—typically with a vertical bar separating the two. A matrix in **row echelon form** is one in which the elements in the diagonal are all ones and the elements under the diagonal are all zeros. A matrix in **reduced row echelon form** is one in which the diagonal consists of ones and all other elements are zeros.

The following row operations can be performed on an augmented matrix without changing the original matrix:

- Any row can be interchanged

- Any row can be replaced by a constant non-zero multiple of that row
- Any row can be replaced by the sum of that row and a constant nonzero multiple of another row

To find the inverse of a matrix, create an augmented matrix with the original matrix in the left side of the bar and the identity matrix on the right side. Convert the left side to reduced row echelon form and the right side will become the inverse matrix. If doing this does not result in the identity matrix on the left side, then the original matrix is singular and has no inverse.

Example 7.0.1

Find the inverse of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We start by creating an augmented matrix with the original matrix on the left and the identity matrix on the right:

$$\left[\begin{array}{ccc|ccc} 3 & 8 & 5 & 1 & 0 & 0 \\ 6 & 4 & 7 & 0 & 1 & 0 \\ 1 & 6 & 9 & 0 & 0 & 1 \end{array} \right].$$

To make things easier, we exchange R_1 and R_2 . Then we add $-6R_1$ to R_2 and $-3R_1$ to R_3 (using the new row names), which gives us

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 9 & 0 & 0 & 1 \\ 0 & -32 & -47 & 0 & 1 & -6 \\ 0 & -10 & -22 & 1 & 0 & -3 \end{array} \right].$$

Next, we divide R_2 by -32 then add $10R_2$ to R_3 to get

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 9 & 0 & 0 & 1 \\ 0 & 1 & \frac{47}{32} & 0 & -\frac{1}{32} & \frac{3}{16} \\ 0 & 0 & -\frac{117}{16} & 1 & -\frac{5}{16} & -\frac{9}{8} \end{array} \right].$$

Next, we multiply R_3 by $-16/117$, add $-6R_2$ to R_1 and add $-47R_3/32$ to R_2 to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{16} & 0 & \frac{3}{16} & -\frac{1}{8} \\ 0 & 1 & 0 & \frac{47}{234} & -\frac{11}{117} & -\frac{1}{26} \\ 0 & 0 & 1 & -\frac{16}{117} & \frac{5}{117} & \frac{2}{13} \end{array} \right].$$

Finally, add $-3R_3/16$ to R_1 to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{39} & \frac{7}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & \frac{47}{234} & -\frac{11}{117} & -\frac{1}{26} \\ 0 & 0 & 1 & -\frac{16}{117} & \frac{5}{117} & \frac{2}{13} \end{array} \right].$$

Now that we have converted it to reduced row echelon form, the left side is the identity matrix and the right side is the inverse matrix of the original one.

The inverse of a matrix can easily be found on a calculator by entering the matrix, selecting the matrix and then selecting the inverse button x^{-1} . To convert everything in the inverse matrix to fractions, use **MATH** > **Frac**.

Matrices can be used to perform operations on complex numbers because the complex number $a + bi$ is isomorphic to the matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

To add, subtract, or multiply 2 complex numbers is the same as doing the same operation with their isomorphic matrix equivalents.

7.1 Coordinate Transformations

Matrices can be used to perform coordinate transformations such as shifting, stretching, and even rotating your graphs in the Cartesian plane.

In general, a linear coordinate transformation is given by a pair of equations of the form

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}$$

We can represent a point (x, y) or a vector to a point $\langle x, y \rangle$ as the column matrix

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then a transformation matrix of the form

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be multiplied by the \mathbf{x} matrix to produce the transformed coordinates given by \mathbf{x}'

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

If we wanted to stretch a graph or a shape on the Cartesian plane vertically by a factor of k , then we know that our transformation equations are

$$\begin{aligned} x' &= x \\ y' &= ky. \end{aligned}$$

In other words, the x coordinates of the **stretched** graph remains the same, but all the y coordinates are doubled. In this case, our transformation matrix would be

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

Similarly, a stretch by a factor of k in the horizontal direction is given by the transformation equation

$$\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}.$$

A **reflection** about an axis is when a graph or shape is flipped about an axis. For a reflection about the x axis, the x coordinates stay the same, but the y coordinates are negated, giving us the transformation matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For a reflection about the y axis, the y coordinates stay the same, but the x coordinates are negated, giving us the transformation matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For a reflection about $y = x$, the x and y coordinates are switched giving us the transformation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A **shear** transformation is when points are shifted in a specific direction by an amount proportional to their distance from a line parallel to that direction. For example, if we have a square sitting on the x -axis and we “sheared” it to the right, the base of the square would remain in the same place, but the top of the square would be shifted to the right so that both sides are no longer vertical. In other words, a parallelogram is a shear transformation of a square. A shear along the x -axis has the form **Shear transformation**

$$\begin{aligned}x' &= x + ky \\y' &= y,\end{aligned}$$

so the transformation matrix is given by

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Similarly, a shear in the y -direction has the transformation matrix

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

The final transformation we’ll look at is the **rotation**. For a graph or shape rotated counterclockwise by an angle θ , the transformation matrix is given by

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

To perform the inverse of a transformation, that is, to undo a transformation, just use the inverse of the transformation matrix.

Chapter 8

Systems of Equations

A system of equations is **consistent** if it has one or more solutions. It is **inconsistent** if it has no solutions.

In a Cartesian coordinate system, the lines of linear equations can either

- Intersect: One solution to the system of equations (consistent and **independent**)
- Be the same: Infinite solutions to the system of equations (consistent and **dependent**)
- Be parallel: No solutions to the system of equations (inconsistent)

There are several different ways of solving systems of equations.

8.1 Substitution

Substitution (also known as elimination) when it comes to a pair of equations simply involves solving one equation for a variable and plugging it into the other.

When dealing with a set of three equations and three unknowns, you have to eliminate one variable at a time using pairs of equations.

Example 8.1.1

Solve the following system of equations by substitution

$$-3x + 2y - 6z = 6$$

$$5x + 7y - 5z = 6$$

$$x + 4y = 8$$

We start with the simplest equation, which is the third one. If there were no simplest equation we would start with any. We solve this equation for x to get $x = 8 - 4y$ and then plug this value into both of the other two equations,

$$-3(8 - 4y) + 2y - 6z = 6$$

$$5(8 - 4y) + 7y - 5z = 6$$

which we solve for y to get

$$y = -\frac{5}{13}z + \frac{34}{13}$$

$$y = \frac{3}{7}z + \frac{15}{7}$$

Equating the two and solving for z gives us $z = 43/74$. Plugging this back into either of the above two equations and solving for y gives us $y = 177/74$. Plugging

both of these back into any of the original three equations and solving for x gives us $x = -58/37$.

8.2 Gauss-Jordan Elimination

To solve a system of equations you can create an augmented matrix of the coefficients of the equations on the left side and the constants of the equations on the right side. The equations must all be in a standard form with variables on the left side and in the same order and the constants on the right side of the equality. A missing variable in an equation is the equivalent of the variable with a coefficient of zero. To solve the system of equations, the coefficient portion of the matrix must be converted to reduced row echelon form, then the solutions will be the right side of the augmented matrix.

Example 8.2.1

Solve the system of equations

$$-3x + 2y - 6z = 6$$

$$5x + 7y - 5z = 6$$

$$x + 4y = 8$$

using Gauss-Jordan elimination.

Our augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 2 & -6 & 6 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & 0 & 8 \end{array} \right].$$

Putting it in reduced row echelon form gives us

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{58}{37} \\ 0 & 1 & 0 & \frac{177}{74} \\ 0 & 0 & 1 & \frac{43}{74} \end{array} \right],$$

which tells us that the solution is

$$x = -\frac{58}{37}, \quad y = \frac{177}{74}, \quad z = \frac{43}{74}.$$

Tip

You can solve a system of equations on your calculator easily with Gauss-Jordan elimination. Just enter your augmented matrix as a regular matrix, then you use **2ND** ► **MATRIX** ► **MATH** ► **rref**(then enter your matrix. This will return your matrix in reduced row echelon form after which the right-most column will be your solution set.

Tip

To solve systems of rational equations with variables in the denominators, make substitutions like $u = 1/x$, $v = 1/y$ and so on, then solve the system of equations for u, v, \dots before doing the back-substitution to return to variables in x, y, \dots

8.3 Inverse Matrix

Another way to solve a system of equations using matrices is to create a matrix of your coefficients and a second (single column) matrix of the constants. Next, you find the inverse of the coefficient matrix and multiply it by the matrix of constants. The result will be a single-column matrix that is your solution set.

On the calculator, if you enter your coefficients into matrix **A** and your constants into matrix **B**, then your solution matrix will be given by $\mathbf{A}^{-1}\mathbf{B}$.

8.4 Cramer's Rule

Cramer's rule uses the determinants of special matrices to solve systems of equations. To use Cramer's rule, the number of equations must equal the number of variables. Cramer's rule for solving a system of equations with three variables is

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

where the equations are

$$\begin{aligned} ax + by &= s \\ cx + dy &= t \end{aligned}$$

and the determinants are

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}, \quad D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}.$$

Notice that D_x is just like D , except that the x coefficients of the system of equations are replaced by the constants of the system of equations. It's the same for D_y . Notice that this method can easily be expanded to a system of three or more equations—it's just more complicated because you have to deal with four 3×3 determinants.

If $D = 0$, then Cramer's rule cannot be used to solve the system and the system must have zero or an infinite number of solutions. If D is zero and at least one of the determinants D_x , D_y , or D_z are not zero, then the system is inconsistent and has no solution. If D is zero and all of the determinants D_x , D_y , or D_z are also zero then the system is dependent and there are infinitely many solutions.

Example 8.4.1

Solve the following system of equations using Cramer's rule

$$\begin{aligned} -3x + 2y - 6z &= 6 \\ 5x + 7y - 5z &= 6 \\ x + 4y &= 8 \end{aligned}$$

Our four determinants are

$$\begin{aligned} D &= \begin{vmatrix} -3 & 2 & -6 \\ 5 & 7 & -5 \\ 1 & 4 & 0 \end{vmatrix} = -148, & D_x &= \begin{vmatrix} 6 & 2 & -6 \\ 6 & 7 & -5 \\ 8 & 4 & 0 \end{vmatrix} = 232 \\ D_y &= \begin{vmatrix} -3 & 6 & -6 \\ 5 & 6 & -5 \\ 1 & 8 & 0 \end{vmatrix} = -354, & D_z &= \begin{vmatrix} -3 & 2 & 6 \\ 5 & 7 & 6 \\ 1 & 4 & 8 \end{vmatrix} = -86 \end{aligned}$$

so our solutions are

$$x = \frac{D_x}{D} = -\frac{58}{37}, \quad y = \frac{D_y}{D} = \frac{177}{74}, \quad z = \frac{D_z}{D} = \frac{43}{74}.$$

8.5 Curve Fitting

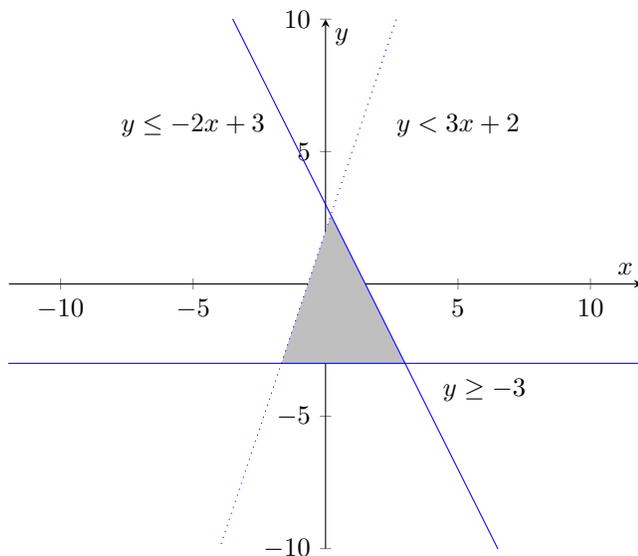
To find the quadratic equation of a curve containing 3 points, build a system of equations in 3 variables by substituting the points (one at a time) into the equation $y = ax^2 + bx + c$. Then solve the system of equations.

8.6 Systems of Inequalities

The solution to a system of inequalities is typically a *region* on the Cartesian plane. To graph a system of inequalities, graph each inequality as a line, then determine from the greater than or less than symbol which side of the line is in the solution set. If the line itself is included in the solution set, graph it as a solid line, otherwise graph it as a dotted line. The solution set to a system of inequalities is the region in each of the solution sets.

The graph shown below is the region that satisfies the system of linear inequalities

$$\begin{aligned}y &< 3x + 2 \\y &\leq -2x + 3 \\y &\geq -3\end{aligned}$$



8.7 Linear Programming

In linear programming, we have an equation that we want to minimize or maximize, which is subject to certain **constraints**. These constraints typically take the form of a system of linear inequalities. By graphing the system of linear inequalities, we obtain a region called the **feasibility region**. One of the corners of this feasibility region will give the maximum or minimum of the function being optimized. To find the optimal value simply test the function with each of the corners of the feasibility region.

Example 8.7.1

Maximize $z = 2x + 3y$ subject to the constraints

$$\begin{aligned}y &\leq x \\y &\leq -2x + 4 \\y &\geq 0\end{aligned}$$

Graphing the system of inequalities (described in the previous section) gives us a feasibility region with corners $(0, 0)$, $(4/3, 4/3)$, and $(2, 0)$. Testing each of these corners in our function gives us

$$\begin{aligned}z &= 2(0) + 3(0) = 0 \\z &= 2\left(\frac{4}{3}\right) + 3\left(\frac{4}{3}\right) = \frac{20}{3} \\z &= 2(2) + 3(0) = 4,\end{aligned}$$

which means the maximum of $z = 20/3$ occurs when $x = 4/3$ and $y = 4/3$.

Example 8.7.2

A technology company produces two kinds of computers—the desktop and the laptop. Production demand means that they must produce at least 100 laptops this month. They only have the materials to produce 100 desktops and 200 laptops. Their contract requires that they ship at least 180 computers this month. If their profit on a desktop is \$150 and their loss on a laptop is \$50, how many of each should they build to maximize profit?

If we let x be the number of laptops made, y be the number of desktops made, and P be the profit, then the function that needs to be maximized is $P = 100x + 150y$. Because they need at least 100 laptops, one constraint is $x \geq 100$. Since they cannot make a negative number of desktops, we can add the constraint $y \geq 0$. Because of their limited amount of materials, two more constraints are $y \leq 100$ and $x \leq 200$. Finally, their contract gives us the final constraint $x + y \geq 180$. In summary, our problem is to maximize $P = -50x + 150y$ subject to the constraints

$$\begin{aligned}100 &\leq x \leq 200 \\0 &\leq y \leq 100 \\y &\geq -x + 180.\end{aligned}$$

Graphing the feasibility region gives us the corners $(100, 80)$, $(100, 100)$, $(200, 100)$, $(200, 0)$, and $(180, 0)$. Testing each of the corners in $P = -50x + 150y$ gives us the maximum $P = \$10,000$ which occurs when $x = 100$ and $y = 100$. In other words, the maximum profit of \$10,000 occurs when 100 laptops and 100 desktops are made.

Chapter 9

Partial Fraction Decomposition

Partial fraction decomposition is when you decompose a rational fraction into the sum or difference of two or more simpler fractions.

Any improper fraction can be reduced to a polynomial plus or minus a smaller proper fraction by using long division.

To perform partial fraction decomposition:

1. First make sure that the rational function is actually a proper fraction. If the degree of the numerator is equal to or larger than the degree of the denominator, use long division to break it into a polynomial plus a proper rational remainder.
2. Completely factor the denominator
3. Determine which “template” to use based on the factors of the denominator. If there are multiple factors of different types, you have to chain multiple templates together

- Non-repeated linear factors:

$$\frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c} + \dots$$

- Repeated linear factors:

$$\frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \dots$$

- Non-repeated irreducible quadratic factors:

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{ax^2+bx+c} + \dots$$

- Repeated irreducible quadratic factors:

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \dots$$

- A rational function containing a combination of these types will result in a combination of types on the other side of the equals sign.
4. After the appropriate template has been constructed, the goal is to determine the values of the constants in the numerators. The ones in the denominators are simply the coefficients of the factors, so for any specific problem these will be known. The first step of this process is to clear the fractions by multiplying both sides of the equals sign by the denominator of the original fraction.
 5. There are several techniques used to work out what the numerator constants are:
 - Substitute convenient numbers for x such that all but one of the constants is canceled out by zeros. Sometimes you can solve for all the constants in this way. However, use this method only in the non-repeated linear factors case.

Tip

A good way to check your partial fraction decompositions at home is to use wxMaxima's partial fraction decomposition function. First, enter your fraction, then highlight it, and go to **Calculus > Partial Fractions**, and select your variable.

- Use a system of equations. This method is guaranteed to solve for the unknowns, but it might not be quite as easy as the method above. First you expand the entire right side, then you gather the terms together to standardize the right side, so to speak, you equate the coefficients. Here's an example: $2x + 1 = (A + B)x + (B - C)$. Then you know that $A + B = 2$ and you can set up a system of linear equations to solve for the unknowns. To do this on a calculator, enter two matrices—one looking like the left side of your system of equations and the other looking like the right side. Now you take the inverse of the first matrix and multiply it by the second matrix. The result is a single column matrix containing the values of your unknowns.
- The fastest way is usually to use a combination of the above techniques. Before expanding the right side, plug in any numbers that will conveniently give you values for the unknowns, and then set up a system of equations for everything. The values you've already figured out may be enough for you to easily solve the system of equations without having to use matrices.

At each step of the process above, you can do a cursory check of your work by plugging your current equation and the original equation into your calculator to see if the graphs match.

Example 9.0.1

Decompose

$$\frac{10x}{x^2 + x - 6}$$

Factoring the denominator and using the template for non-repeated linear factors, we get

$$\frac{10x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiplying both sides by $(x+3)(x-2)$ gives us

$$10x = A(x-2) + B(x-2)$$

If we let $x = 2$ and solve for B , we get $B = 4$. If we let $x = -3$ and solve for A , we get $A = 6$, so our decomposed fraction is

$$\frac{10x}{x^2 + x - 6} = \frac{6}{x+3} + \frac{4}{x-2}$$

Example 9.0.2

Decompose

$$\frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{x^5 + 2x^4 + 4x^3 + 8x^2 + 4x + 8}$$

Graphing the denominator suggests a factor of $x + 2$ in the denominator and long division of the denominator by $x + 2$ gives a remainder of zero, confirming this. The quotient resulting from the long division is $x^4 + 4x^2 + 4$, so

$$\frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{x^5 + 2x^4 + 4x^3 + 8x^2 + 4x + 8} = \frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{(x+2)(x^4 + 4x^2 + 4)}$$

Since $x^4 + 4x^2 + 4$ is a quadratic-like polynomial, we let $u = x^2$ and use the quadratic formula on $u^2 + 4u + 4$ to get $u = -2$, which implies $x^2 = -2$. Since we

can't factor this further, our simplified fraction is

$$\frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{x^5 + 2x^4 + 4x^3 + 8x^2 + 4x + 8} = \frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{(x+2)(x^2+2)^2}.$$

Applying the template for repeated quadratic factors and chaining it with the template for non-repeated linear factors gives us

$$\frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{(x+2)(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{E}{x+2}.$$

Multiplying both sides by the denominator of the left side to clear the fractions gives us

$$6x^4 + 4x^3 + 29x^2 + 18x + 36 = (Ax+B)(x^2+2)(x+2) + (Cx+D)(x+2) + E(x^2+2)^2.$$

Expanding the right side and gathering *like* terms gives us

$$6x^4 + 4x^3 + 29x^2 + 18x + 36 = (A+E)x^4 + (2A+B)x^3 + (2A+2B+C+4E)x^2 + (4A+2B+2C+D)x + (4B+2D+4E).$$

By comparing with the coefficients on the left side we can build a system of equations. For example, looking at the coefficients of x^4 on both sides we see that $6 = A + E$. Using the inverse matrix method to solve the system of five equations give us

$$\begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 2 & 2 & 1 & & 4 \\ 4 & 2 & 2 & 1 & \\ & 4 & & 2 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} 6 \\ 4 \\ 29 \\ 18 \\ 36 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix},$$

and so our solution is

$$\frac{6x^4 + 4x^3 + 29x^2 + 18x + 36}{(x+2)(x^2+2)^2} = \frac{x+2}{x^2+2} + \frac{3x+4}{(x^2+2)^2} + \frac{5}{x+2}.$$

Chapter 10

Sequence, Series, and Mathematical Induction

10.1 Sequences and Series

A **sequence** is a function whose domain is the set of positive integers.

When representing sequence functions, the variable n is used rather than x . This is to demonstrate that the inputs are positive integers.

When a formula for the n th (or general) term is known, the sequence is represented by placing braces around the formula for the n th term. For example

$$\{b_n\} = \left\{ \left(\frac{1}{2} \right)^n \right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

Sequences can be graphed on a calculator, but the graph will be a set of points rather than a line or a curve.

A graphing calculator can be used to generate the first terms of a sequence. Change the mode to **seq** then go to **2nd > List > OPS > seq(**. Then enter the formula for the n th term, enter a comma, enter the variable, enter a comma, enter the smallest input, enter a comma, enter the largest input, and lastly enter a parenthesis. Like so: **seq((1/2)ⁿ, n, 1, 10)**. Then go to **Math > Frac** to convert the shown terms to fractions instead of decimals. You may have to scroll right to see all of the terms.

You can also graph sequences on the calculator. Make sure it's in sequence mode, first. For the graph, enter the formula for the n th term at **u(n)=** and then enter the largest input at **u(nMin)**.

A sequence is defined recursively when it's written such that a term depends on previous terms. The Fibonacci sequence, for example, is a **recursive sequence**.

Summation notation is used when you want to find the sum of the first n terms in a sequence

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n.$$

The Greek letter Sigma is simply an instruction to sum all the terms. The part to the right of sigma is the formula for the sequence, the k is the index and in this case the summation starts at an index of 1 and ends at an index of n . In words—the sum of a_k from $k = 1$ to $k = n$. The index of summation doesn't always start at 1 or end at n . Also, any letter may be used to represent the index.

Here are some properties of sequences that need to be memorized. If $\{a_n\}$ and $\{b_n\}$

are sequences and c is a real number then

$$\begin{aligned}\sum_{k=1}^n (ca_k) &= c \sum_{k=1}^n a_k \\ \sum_{k=1}^n (a_k + b_k) &= \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \\ \sum_{k=1}^n (a_k - b_k) &= \sum_{k=1}^n a_k - \sum_{k=1}^n b_k \\ \sum_{k=j+1}^n a_k &= \sum_{k=1}^n a_k - \sum_{k=1}^j a_k \text{ if } 0 < j < n\end{aligned}$$

Here are formulas for the sum of the first n terms of several different sequences

$$\begin{aligned}\sum_{k=1}^n c &= cn \\ \sum_{k=1}^n k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^n k^3 &= \left[\frac{n(n+1)}{2} \right]^2\end{aligned}$$

When the difference between consecutive terms of a sequence is always the same, the sequence is called an **arithmetic sequence**. The common difference between consecutive terms of an arithmetic sequence is

$$a_n - a_{n-1}.$$

The n th formula for any arithmetic sequence whose first term is a_1 and whose common difference is d is

$$a_n = a_1 + (n-1)d.$$

The recursive formula for the same is

$$a_1 = d, a_n = a_{n-1} + d.$$

Example 10.1.1

Find the 47th term of the arithmetic sequence for which the 10th term is 68 and the 20th term is 118.

We are given $a_{10} = 68$ and $a_{20} = 118$, using the formula $a_n = a_1 + (n-1)d$, we can set up the system of linear equations

$$\begin{aligned}68 &= a_1 + (10-1)d \\ 118 &= a_1 + (20-1)d,\end{aligned}$$

that we solve to find $a_1 = 23$ and $d = 5$. Now, we just plug this back into the formula to get

$$a_{47} = 23 + (47-1)5 = 253.$$

The **partial sum** of a sequence is the sum of some number of terms, usually the first n terms of a sequence. It is denoted S . The partial sum of the first n terms of an arithmetic sequence can be calculated using either of the following formulas

$$S = \frac{n}{2} [2a_1 + (n-1)d]$$

$$S = \frac{n}{2} (a_1 + a_n).$$

The first formula is useful when you know the first term and the common difference. The second formula is useful when you know the first and n th terms of the sequence. In both cases, you have to know the number of terms you're summing n .

Example 10.1.2

Find the sum of

$$2 + 4 + 6 + \cdots + 98 + 100.$$

We are given the sum of the first n terms of an arithmetic sequence with common difference $d = 2$. Examining this partial sum, we see that there are $n = 50$ terms. Using the first formula given above, we get

$$S = \frac{50}{2} [2(2) + (50-1)2] = 2550.$$

Using the second formula given above, gives us the same partial sum

$$S = \frac{50}{2} (2 + 100) = 2550.$$

When given two elements of an arithmetic sequence, you can solve for the first element and the difference variable by creating a system of two linear equations in two variables using the formula for the n th term of an arithmetic sequence.

A sequence is a **geometric sequence** if the ratio of successive terms is always the same nonzero constant. Instead of a common difference, geometric sequences have **common ratios**.

A geometric sequence can be defined recursively as

$$a_1 = a, \quad a_n = ra_{n-1},$$

where r is the **common ratio** between consecutive terms.

The formula for the n th term of a geometric sequence is

$$a_n = a_1 r^{n-1}.$$

The sum of the first n terms of a geometric sequence is calculated as

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}.$$

If the sum of an infinite geometric series converges to a specific number as n gets larger, the series is said to be **convergent**. Otherwise, the series is **divergent**. If the common ratio is strictly between -1 and 1 , that is $|r| < 1$, then the infinite geometric series is convergent and its sum is

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}.$$

10.2 Annuity

If P is the deposit at the end of each payment period for an annuity paying i percent interest per pay period then the amount A of the annuity after n deposits is given by

$$A = P \frac{(1+i)^n - 1}{i}.$$

To figure out i when the payment period is not a year, divide the percent interest by the number of payment periods in a year.

10.3 Mathematical Induction

Mathematical induction is a method for proving that statements involving natural numbers are true for all natural numbers.

The theorem of mathematical induction states that if a statement about natural numbers satisfies two specific conditions, then that statement is true for all natural numbers. The conditions are

1. The statement is true for the number 1
2. If the statement is true for some natural number k then it is also true for the natural number $k + 1$.

Showing something true for the number 1 is trivial. The hard part is demonstrating condition number 2. First substitute k with $k + 1$ and then replace a portion of one side of the equation with one side of the equation involving just k .

Example 10.3.1

Using mathematical induction, prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

First we prove it for the number one

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} = 1.$$

Now we have to prove that if the statement is true for a natural number k then it must necessarily also be true for the natural number $k + 1$. In other words, if

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6},$$

then it follows that

$$1^2 + 2^2 + 3^2 + \cdots + [k+1]^2 = \frac{[k+1]([k+1]+1)(2[k+1]+1)}{6}.$$

To show this, we rewrite the second one in terms of the first one by adding $[k+1]^2$ to both sides

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + [k+1]^2 = \frac{k(k+1)(2k+1)}{6} + [k+1]^2.$$

To make it easier, we'll make the substitution $n = k + 1$, then we only need to

convert the right side into the form of the original equation

$$\begin{aligned}
 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 &= \frac{(n-1)n(2[n-1] + 1)}{6} + n^2 \\
 &= \frac{(n-1)n(2n-1) + 6n^2}{6} \\
 &= \frac{2n^3 + 3n^2 + n}{6} \\
 &= \frac{n(2n^2 + 3n + 1)}{6} \\
 &= \frac{n(n+1)(2n+1)}{6} \quad \blacksquare
 \end{aligned}$$

10.4 Binomial Theorem

The **binomial theorem** is a formula used to expand binomials which have the form $(x + a)^n$. The formula is

$$(x + a)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} a^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The notation $\binom{n}{k}$ is called “ n choose k ”.

In addition to using the formula given above using factorials, you can also calculate $\binom{n}{k}$ using **Pascal’s triangle**. The value is found at the intersection of the n th row and the k th diagonal. Remember that the row and diagonal indices of Pascal’s triangle start at 0.

You can also calculate $\binom{n}{k}$ on a calculator. Use the **nCr** function, which stands for “ n choose r ”, under the **MATH > PRB** menu. For example, to calculate $\binom{100}{10}$, you would enter **100 nCr 10** into the calculator.

Remember that $0! = 1$.

An expanded form of the binomial theorem is shown here

$$(x + a)^n = \binom{n}{0} x^n a^0 + \binom{n}{1} x^{n-1} a^1 + \binom{n}{2} x^{n-2} a^2 + \cdots + \binom{n}{n} x^0 a^n.$$

Example 10.4.1

Expand the binomial $(x - 2)^3$ using the binomial theorem.

$$\begin{aligned}
 (x - 2)^3 &= \binom{3}{0} x^3 (-2)^0 + \binom{3}{1} x^2 (-2)^1 + \binom{3}{2} x^1 (-2)^2 + \binom{3}{3} x^0 (-2)^3 \\
 &= \binom{3}{0} x^3 - 2 \binom{3}{1} x^2 + 4 \binom{3}{2} x - 8 \binom{3}{3} \\
 &= (1)x^3 - 2(3)x^2 + 4(3)x - 8(1) \\
 &= x^3 - 6x^2 + 12x - 8.
 \end{aligned}$$

Here is an illustration of the first several rows of Pascal’s triangle:

$n = 0:$				1												
$n = 1:$				1		1										
$n = 2:$				1		2		1								
$n = 3:$				1		3		3		1						
$n = 4:$				1		4		6		4		1				
$n = 5:$				1		5		10		10		5		1		
$n = 6:$				1		6		15		20		15		6		1

Chapter 11

Polar Coordinates

In the Cartesian coordinate system, a point is represented by (x, y) . In **polar coordinates** a point is represented by a distance r from the origin and θ from the positive x -axis, (r, θ) . The distance r can be thought of as the length of the vector from the origin to the point. If r is negative then the specified point is r units from the pole in the opposite direction of theta. Unlike for Cartesian coordinates, a point given in polar coordinates is not unique. This is because θ can also be any multiple of 360° .

In this convention, the origin is called the **pole** and the positive x -axis is called the **polar axis**.

To convert between polar and rectangle coordinates, draw the point using both coordinate systems on the same plane and just use the Pythagorean theorem and/or the following relationships to do the conversions.

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1} \left(\frac{y}{x} \right).\end{aligned}$$

To convert between polar and rectangular coordinates using a calculator, use the functions found under the Angle menu.

Equations in polar coordinates use r and θ instead of x and y . To convert between the two, use the same relationships (Pythagorean theorem and/or trig relations) that are used to convert between polar and rectangular coordinates.

To graph equations given in polar coordinates, we can make a table of values of θ , x , and y . Choosing convenient values of θ and evaluating the corresponding x and y values allows us to plot the x and y values. We can also eliminate the parameter θ by converting the equation in polar coordinates to one written in Cartesian coordinates.

Example 11.0.1

Convert the equation

$$r^2 = 4 \sin \theta,$$

written in polar coordinates to one written in Cartesian coordinates.

To convert the equation, we use the relations

$$\begin{aligned}r^2 &= x^2 + y^2 \\ y &= r \sin \theta \implies \frac{r}{\sqrt{x^2 + y^2}} = \sin \theta,\end{aligned}$$

to make the substitutions $(x^2 + y^2) = 4y/\sqrt{x^2 + y^2}$. Simplifying, we get

$$(x^2 + y^2)^{\frac{3}{2}} = 4y.$$

Example 11.0.2

The equation

$$x^2 + y^2 = 3x,$$

is written in Cartesian coordinates. Convert it to an equation in polar coordinates.

We use the fact that

$$x = r \cos \theta$$

$$y = r \sin \theta$$

to make the substitutions in the given equation to get $(r \cos \theta)^2 + (r \sin \theta)^2 = 3(r \cos \theta)$. Simplifying gives us

$$r^2 = 3r \cos \theta.$$

To graph a polar equation in a polar coordinate system, just make a table of values for (r, θ) using convenient values of θ such as 30° , 45° , 60° and so on.

You can graph polar equations on a calculator by setting the function mode to pol.

Another way to graph is to substitute x and y for θ and r , draw that graph, then draw the polar graph by deducing the movement relationships between the two graphs.

In math, there are always multiple ways of solving a problem. Solve it one way and check it the other way. If you can't think of the common method of solving the problem, try to think of alternate ways—perhaps using the advanced functions of your graphing calculator. Don't remember the rules for graphing an equation? You can always just make a table of values or do it on the calculator.

Chapter 12

Complex Numbers

Remember that division of complex numbers requires using complex conjugates.

The complex number $x + yi$ can be represented as a vector ending at (x, y) on a 2-dimensional complex plane with a horizontal axis of real numbers and a vertical axis of imaginary numbers. This coordinate system designed for complex numbers is called the **Argand plane**.

The **absolute value** or **modulus** of a complex number is the distance from the origin to the end of the vector. This value is easily found using the Pythagorean theorem.

The **argument** of a complex number denoted as $\arg(z)$ is the smallest positive angle theta between the vector and the positive real axis. This angle can be found using basic trigonometry.

This gives us a new method of representing complex numbers

$$x + yi = r(\cos \theta + i \sin \theta) = r \text{ cis } \theta.$$

To convert a complex number from **standard form** to **trigonometric form** use the Pythagorean theorem and a trigonometric function

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left(\frac{y}{x} \right).$$

To convert from trigonometric form to standard form use

$$x = r \cos \theta, \quad y = r \sin \theta.$$

These relations can be understood by graphing the the complex number as a point on the Argand plane and then noting that it can be represented by (x, y) and/or by (r, θ) .

To convert from trigonometric form to standard form on a TI-84 calculator, you can use **P ► Rx**(r, θ) from the **ANGLE** button to get the x value of the complex number and use **P ► Ry**(r, θ) to get the y value.

Similarly, to convert from standard form to trigonometric form on a TI-84 calculator, you can use **R ► Pr**(x, y) from the **ANGLE** button to get the r value of the complex number and use **R ► Pθ**(x, y) to get the angle.

To multiply two complex numbers in trigonometric form, multiply their moduli and add their arguments:

$$(r_1 \text{ cis } \theta_1) \cdot (r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2).$$

De Moivre's Theorem is a useful formula for finding roots and powers of complex numbers. It states that

$$z^n = r^n \text{ cis } (n\theta)$$
$$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta).$$

Example 12.0.1

Simplify

$$\left(\sqrt{2} \operatorname{cis} \frac{7\pi}{24}\right)^8.$$

Using De Moivre's theorem, we get

$$\left(\sqrt{2} \operatorname{cis} \frac{7\pi}{24}\right)^8 = (\sqrt{2})^8 \operatorname{cis} 8 \cdot \frac{7\pi}{24} = 16 \operatorname{cis} \frac{7\pi}{3} = 16 \operatorname{cis} \frac{\pi}{3}.$$

To divide two complex numbers in trigonometric form, divide their moduli and subtract their arguments

$$\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2).$$

Every complex number has exactly n distinct n th roots. To find the n , n th roots of a complex number

$$z^{\frac{1}{n}} = \operatorname{cis} \left(\frac{\theta}{n} + \frac{360^\circ}{n} k \right), \text{ where } k = 0, 1, 2, \dots, n-1.$$

The root corresponding to $k = 1$ is called the principal n th root of the complex number. Using a calculator will only give the principal n th root of a complex number.

Example 12.0.2

Find the three cube roots of $32\sqrt{3} + 32i$.

Converting to trig form gives us $32\sqrt{3} + 32i = 64 \operatorname{cis} 30^\circ$. Then using the formula given above, we have that $n = 3$ for cube roots, so the roots are

$$4 \operatorname{cis} \left(\frac{30^\circ}{3} + \frac{360^\circ}{3} [0] \right) = 4 \operatorname{cis} 10^\circ$$

$$4 \operatorname{cis} \left(\frac{30^\circ}{3} + \frac{360^\circ}{3} [1] \right) = 4 \operatorname{cis} 130^\circ$$

$$4 \operatorname{cis} \left(\frac{30^\circ}{3} + \frac{360^\circ}{3} [2] \right) = 4 \operatorname{cis} 250^\circ$$

Note: The n , n th roots in trigonometric form are distributed evenly (graphically). In other words, the 4, 4th roots of a complex number will have displayed graphically as 4 vectors that are 90 degrees apart.

Chapter 13

Parametric Equations

Parametric equations show x and y as functions of a third variable

$$x = g(t), \quad y = h(t)$$

There are several ways to graph parametric equations. The simplest, but slow way is to make a table with columns for t , x , and y . Choose different values of t , and calculate the corresponding values for x and y and plot the x and y coordinates on the Cartesian plane. When graphing parametric equations, keep in mind that the graph has direction—it is going in the direction of increasing t and you have to add arrows to your graph to indicate this.

A second way to graph parametric equations is to eliminate the parameter so that you can work only in terms of x and y . You cannot always do this.

Parametric equations can be graphed on a calculator by setting the mode to **PAR** instead of **FUNC**.

To **eliminate the parameter** from a pair of parametric equations you can often solve for the parameter t , then set the two results equal to each other and solve for y . If you eliminate the parameter and choose to graph using only the equation in terms of x and y , your result has lost the direction information. To find the direction of the graph you'll have to plug a couple values of t into the original parametric equations.

Example 13.0.1

Eliminate the parameter t from the set of parametric equations

$$x = t^2 + 1, \quad y = 2t - 1.$$

Solving each equation for t gives us $t = \pm\sqrt{x-1}$ and $t/2 + 1/2$. Setting these equal to each other and solving for y gives us the pair of equations

$$y = 2\sqrt{x-1} - 1$$

$$y = -2\sqrt{x-1} - 1.$$

Notice that eliminating the parameter gives us a pair of equations that describes a parabola whereas the actual set of parametric equations only describes half a parabola.

Example 13.0.2

Eliminate the parameter t from the set of parametric equations

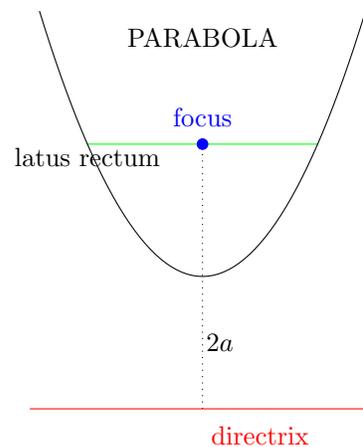
$$x = 3t - 2, \quad y = 2t + 1.$$

Solving both of them for t , equating the two, and solving for y gives us the equation of a line

$$y = \frac{2}{3}x + \frac{7}{3}.$$

Chapter 14

Conic Sections



Through every point on the surface of a right-circular cone, there is a straight line. These lines are called **generators**. The cone consists of two parts called **nappes**, which are separated by the vertex.

A **hyperbola** is a plane that intersects both nappes. A **parabola** intersects only one nappe and the plane of a parabola is parallel to one and only one generator of the cone. An **ellipse** slices through one nappe.

If the plane of the conic section contains the cone's vertex, that plane is a **degenerate conic**.

14.1 Parabolas

A **parabola** is the set of all points in the plane that are the same distance from a fixed point—the **focus**, as they are from a fixed line—the **directrix**.

The **vertex** of a parabola is exactly midway between its focus and directrix.

The general equations of parabolas with axes of symmetry that lie along the x or y axes are shown below. The constant a is the distance from the vertex to either the focus or the directrix.

$$y^2 = 4ax \quad \text{opens to the right}$$

$$y^2 = -4ax \quad \text{opens to the left}$$

$$x^2 = 4ay \quad \text{opens upward}$$

$$x^2 = -4ay \quad \text{opens downward}$$

Notice that if you solve for y , for the parabolas that open to the left or right, you get the equations for radical functions on both the positive and negative side of the x -axis.

For parabolas that open upward or downward, you simply get quadratic equations, which you know are parabolas.

To find the focus and directrix of a parabola, put it in the form of one of the above equations, then you can use the fact that the distance between the vertex and both the focus and the directrix is x . In other words, the distance between the focus and the directrix is $2a$.

For parabolas with vertexes that are shifted up or down and left or right but still have axes of symmetry that are parallel to the x or y -axes, the equations stay the same except that $(x - h)$ is substituted for x and $(y - k)$ is substituted for y . This gives the equation for a parabola with vertex at (h, k) . Use the idea of transformations to remember how to derive these equations.

The equations of parabolas can be derived from the knowledge that for any point P with coordinates (x, y) , the distance from P to the directrix is equal to the distance from P to the focus.

The line segment running through the focus and joining the points on the parabola directly above and below the focus is called the **latus rectum**. Use the latus rectum to graph the opening of parabolas. The distance from the focus to either endpoint of the latus rectum is $2a$.

Polynomial equations define parabolas whenever one variable is quadratic and the other variable is linear.

Remember that for a quadratic equation in the **standard form** form $f(x) = ax^2 + bx + c$, the vertex occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

Note: This is not the same a as the one used in the equations above. You can also find the vertex by putting the equation into **vertex form**

$$y = a(x - h)^2 + k,$$

then the vertex will be at (h, k) . To convert from standard form to vertex form, you need to **complete the square**.

In general, to complete the square, we follow these steps:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= (ax^2 + bx) + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

Example 14.1.1

Put

$$y = 2x^2 - x - 6,$$

into vertex form.

We start by separating the terms containing x from the constant

$$y = (2x^2 - x) - 6.$$

Next, we factor out the leading coefficient to get

$$y = 2\left(x^2 - \frac{1}{2}x\right) - 6.$$

Now we need to add a term inside the parentheses that will make the expression on the inside a perfect square. To do that, we take the coefficient on x (in our case it's $1/2$), divide it by 2, square it, and add it inside the parentheses. However, in order to not change the equation, we need to subtract the same amount outside the parentheses. Since the amount inside is multiplied by a factor of 2, we also need to multiply the part that we're subtracting from the outside by 2.

$$y = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - 6 - \frac{2}{16}.$$

Simplifying, we get

$$y = 2\left(x - \frac{1}{4}\right)^2 - \frac{49}{8},$$

so the vertex of this parabola is at $(1/4, -49/8)$.

Example 14.1.2

Construct a parabola such that the focus lies at the point $(5, 3)$ and the directrix is the line $y = 1$.

Drawing the focus and the directrix on the Cartesian plane, we see that the parabola must be an upward opening parabola, so it must have the form $x^2 = 4ay$. Since the vertex of a parabola occurs exactly halfway between its focus and its directrix, we know that its vertex must occur at the point $(5, 2)$. Since the parabola's vertex is not at the origin, we rewrite its equation as $(x-5)^2 = 4a(y-2)$. Since the distance from the vertex to the focus and the directrix is 1, we have that $a = 1$. Plugging this in and solving for y gives us our equation for the parabola

$$y = \frac{1}{4}(x - 5)^2 + 2.$$

Example 14.1.3

Where is the focus and the directrix for the parabola

$$y = x^2 + 2x - 4?$$

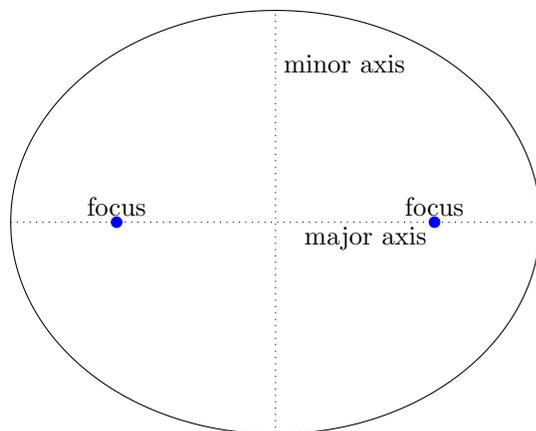
We know that this is a parabola with vertical symmetry (as opposed to horizontal) since it is a quadratic equation. Since the leading coefficient is positive, we know that it opens up vertically. Using $x = -b/(2a)$ and plugging this back in, we find that the vertex of this parabola occurs at $(-1, -5)$. Since this equation has the form $x^2 = 4ay$, we replace x and y with $x + 1$ and $y + 5$ since the vertex is not at the origin

$$(x + 1)^2 = 4a(y + 5).$$

Solving this equation for y , and setting it equal to our original equation allows us to solve for a , getting $a = 1/4$. Since we know that the focus is a above the vertex and the directrix is a below the vertex, we have that

Focus:	$(-1, -4.75)$
Directrix:	$y = -5.25$

When finding the focus of an off-origin parabola, always remember to add or subtract h or k .



14.2 Ellipses

An **ellipse** is the set of all points in the plane, the sum of whose distances from two fixed points called the **foci**, is a constant.

The line containing the foci is called the **major axis**. The **semi-major axis** is half of the major axis. The line perpendicular to that is the **minor axis**. The **semi-minor axis** is half of the minor axis. The midpoint between the foci is the **center**. Where the major axis intersects the ellipse, is called **vertices**. The length of the major axis is the distance between vertices.

An ellipse where both foci are the same point is a circle. An ellipse where one point is stationary and the other point is arbitrarily far away, tends to a parabola. In other words, a parabola can be thought of as looking at one focus (and end) of an ellipse where the other focus is infinitely far away in the direction of the open end of the parabola.

The equation of an ellipse centered at the origin, with the major axis along the x -axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = a^2 - c^2$. The constant c is the distance between the center and either focus.

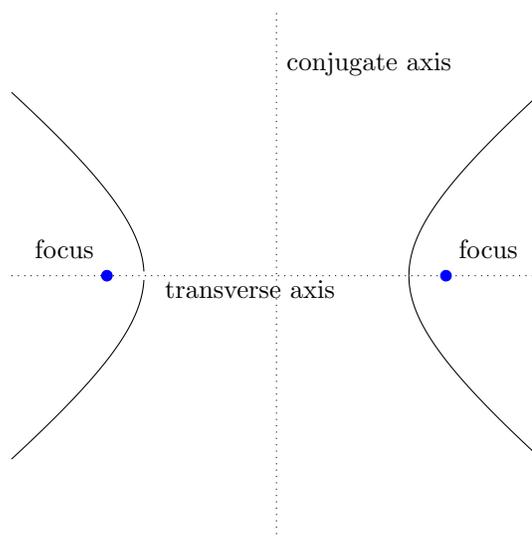
The equation of an ellipse centered at the origin, with the major axis along the y -axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = a^2 - c^2$. The constant c is the distance between the center and either focus.

Notice that a^2 is always greater than b^2 since a is defined as the length of the semi-major axis and b is defined as the length of the semi-minor axis. Therefore, if the larger number is under the x term, then the major axis is horizontal. If the larger number is under the y term, the major axis is vertical.

For ellipses that are translated horizontally and vertically, the equations are the same except x is replaced with $x - h$ and y is replaced with $y - k$.



14.3 Hyperbolas

A **hyperbola** is like the opposite of an ellipse. Instead of a point being the sum of distances to two foci, the points on a hyperbola are the difference of distances to two foci. The line containing the foci is the **transverse axis**. The **center** is halfway between the foci, and the line going through the center and perpendicular to the transverse axis is the **conjugate axis**. Each curve is called a branch.

The equation of a hyperbola centered on the origin with the transverse axis along the x -axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = c^2 - a^2$.

The equation of a hyperbola centered on the origin with the transverse axis along the y -axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = c^2 - a^2$.

The distance from the center to a focus is c which is calculated as $c = \pm\sqrt{b^2 + a^2}$. Notice that the foci are still “inside” of each parabola.

Every hyperbola has two asymptotes. The horizontal hyperbola has the asymptotes $y = bx/a$ and $y = -bx/a$. The vertical hyperbola has the asymptotes $y = ax/b$ and $y = -ax/b$. An easy way to derive the asymptotes and to help graph the hyperbola is to construct a rectangle with the points a and b where they lie on the hyperbola’s axes. In other words, for a hyperbola centered on the origin, you would construct a rectangle with the points $(-a, 0)$, $(a, 0)$, $(-b, 0)$, and $(b, 0)$. The asymptotes are the lines that connect the diagonals and continue in either direction. Instead of remembering the formulas for finding the asymptotes, simply draw the rectangle based on the hyperbola’s equation, then figure out the equations for the asymptotes using slope, etc.

For hyperbolas not centered on the origin, the equations are the same except x is replaced with $x - h$ and y is replaced with $y - k$.

14.4 General Form

The general form of the equation of a conic section with axes parallel to the coordinate axes is

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

To put it into standard form so that it's easy to find the center and other characteristics of the conic section, we complete the square for the x 's and complete the square for the y 's.

Example 14.4.1

Identify the conic section and its characteristics given the equation

$$9x^2 + 16y^2 + 18x - 96y + 9 = 0.$$

Rearranging it and completing the square for each of the variables to rewrite the equation in standard form gives us

$$\begin{aligned} (9x^2 + 18x) + (16y^2 - 96y) + 9 &= 0 \\ 9(x^2 + 2x) + 16(y^2 - 6y) + 9 &= 0 \\ 9(x^2 + 2x + 1) - 9(1) + 16(y^2 - 6y + 9) - 16(9) + 9 &= 0 \\ 9(x + 1)^2 + 16(y - 3)^2 &= 144 \\ \frac{(x + 1)^2}{16} + \frac{(y - 3)^2}{9} &= 1 \end{aligned}$$

This is the equation of an ellipse centered at $(-1, 3)$ with a semi-major axis 4 and a semi-minor axis 3.

What if the axes of the conic section are not parallel to the coordinate axes? In that case, the general form of the equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Notice the extra Bxy term. The coefficient B is zero when the axes of the conic section are parallel to the axes of the coordinate system. From the equation, we can determine the kind of conic section that we're dealing with:

- If $B^2 - 4AC < 0$, it is an ellipse
- If $B^2 - 4AC = 0$, it is a parabola
- If $B^2 - 4AC > 0$, it is a hyperbola

The quantity $B^2 - 4AC$ is called the **discriminant**. Sometimes, the result is a **degenerate conic**, such as a line or a point.

In order to write a rotated conic section in standard form so that we can find its center and its axes, we need to get rid of the Bxy term by creating a new rotated coordinate system so that the axes of the conic section are parallel to the new coordinate system. To find the angle of rotation, we solve

$$\tan(2\theta) = \frac{B}{A - C},$$

for θ . Next, our coordinate transformations are

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta \\ y &= x' \sin \theta + y' \cos \theta \\ x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta, \end{aligned}$$

where the prime symbol means it's part of the rotated coordinate system. After transforming everything, our equation will be in the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0,$$

after which we can rewrite it in standard form.

So to find the center of a rotated conic section, for example, we find the angle of rotation, and then transform the given equation into an equation on the new coordinate system by applying the coordinate transformations. Then, we rewrite this new equation in standard form to find the center of the conic section on the rotated coordinate system (x', y') . Finally, using the coordinate transformations again, we convert this point into one in the original coordinate system.

During the transformation, $B^2 - 4AC = B'^2 - 4A'C'$, so this can be used to determine what type of conic section you're dealing with before you do the transformation.

Example 14.4.2

Rotate the coordinate system to identify the characteristics of the conic section given by

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 64 = 0.$$

First, we calculate $B^2 - 4AC < 0$ to determine that we're dealing with an ellipse. Next, we determine the angle that it is rotated through

$$\tan(2\theta) = \frac{-6\sqrt{3}}{7-13} \implies \theta = \frac{1}{2} \tan^{-1} \sqrt{3} = 30^\circ.$$

Next, we perform the coordinate transformations to get our new variables

$$\begin{aligned} x &= x' \cos 30^\circ - y' \sin 30^\circ = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \\ y &= x' \sin 30^\circ + y' \cos 30^\circ = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'. \end{aligned}$$

Plugging these back into our equation and simplifying gives us

$$\begin{aligned} 0 &= 7 \left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \right)^2 - 6\sqrt{3} \left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \right) \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \right) \\ &\quad + 13 \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \right)^2 - 64 \\ 0 &= 4x'^2 + 16y'^2 - 64 \\ 1 &= \frac{x'^2}{16} + \frac{y'^2}{4}. \end{aligned}$$

This tells us that our ellipse is centered at the origin (in both coordinate systems), its semi-major axis is 4, its semi-minor axis is 2, and we already knew that it is rotated so that its semi-major axis lies along the 30° angle relative to the original coordinate system.

14.5 Parametric Form

The conic sections can also be written in parametric equations.

The parametric equations for a parabola $(x - h)^2 = 4a(y - k)$ with vertex at (h, k) are

$$x = 2at + h, \quad y = at^2 + k.$$

Note: To get the complete graph on a calculator, you may have to enter some negative value for **Tmin**.

Example 14.5.1

Parametrize the parabola

$$y = 4x^2 - 24x + 34.$$

Completing the square and putting it in the specified form, we get

$$(x - 3)^2 = \frac{1}{4}(y + 2),$$

which indicates a parabola with vertex at $(3, -2)$. Since $4a = 1/4$ implies $a = 1/16$, our parametric equations are

$$x = \frac{t}{8} + 3, \quad y = \frac{t^2}{16} - 2.$$

The standard equation of an ellipse centered at (h, k) with the major axis along the x -axis is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = a^2 - c^2$. The constant c is the distance between the center and either focus. The parametric form is

$$x = a \cos t + h, \quad y = b \sin t + k.$$

For ellipses with the major axis along the y -axis, the standard equation is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1,$$

The parametric form is

$$x = b \sin t + h, \quad y = a \cos t + k.$$

The standard equation of a hyperbola centered at (h, k) with the transverse axis along the x -axis is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1,$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis and $b^2 = c^2 - a^2$. The parametric form is

$$x = a \sec t + h, \quad y = b \tan t + k.$$

For hyperbolas with the transverse axis along the y -axis, the standard equation is

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

The parametric form is

$$x = b \tan t + h, \quad y = a \sec t + k.$$

By plugging these parametric equations into the standard equation and simplifying using trigonometric identities, it is easy to show that they are true.

14.6 Polar Form

The equations for conic sections can also be written in polar form. Here, we define conic sections in a slightly different way.

We define a conic section as a set of points in which the distance between any point on the conic and a fixed focus point is related to the distance from that point to a fixed line called the **directrix** by a constant ratio e called the **eccentricity**.

The eccentricity defines the type of conic section that you're dealing with:

- If $0 < e < 1$, it is an ellipse
- If $e = 1$, it is a parabola
- If $e > 1$, it is a hyperbola

For conic sections in polar form, the focus (or one of them) of the conic section is at the pole (i.e. the origin). We let p be the distance between the pole (i.e. the focus) and the directrix.

If the directrix is horizontal and p units above the horizontal axis, then the main axis of the conic section is vertical (on the line $\theta = \pi/2$ and the equation is

$$r = \frac{ep}{1 + e \sin \theta}.$$

If the directrix is horizontal and p units below the horizontal axis, then the main axis of the conic section is vertical (on the line $\theta = \pi/2$ and the equation is

$$r = \frac{ep}{1 - e \sin \theta}.$$

If the directrix is vertical and p units to the right of the pole, then the main axis of the conic section is horizontal (on the line $\theta = 0$ and the equation is

$$r = \frac{ep}{1 + e \cos \theta}.$$

If the directrix is vertical and p units to the left of the pole, then the main axis of the conic section is horizontal (on the line $\theta = 0$ and the equation is

$$r = \frac{ep}{1 - e \cos \theta}.$$

To analyze an equation for a conic section in polar form, first ensure that it is in the form shown above. You may have to do some rearranging. Then determine the type of conic by determining the value of e , which can be found as the coefficient on the trigonometric function. Next, you can determine the directrix from the fact that the numerator equals ep . You can now graph the conic section.

If a constant term is added to the argument of the trigonometric function then the conic section is rotated.

Example 14.6.1

Discuss the characteristics of the conic section

$$r = \frac{1}{1 + \frac{1}{2} \cos \left(\theta + \frac{\pi}{3} \right)}.$$

The form of the equation tells us that the main axis of the conic section is horizontal and the directrix is vertical. However, the $\pi/3$ term in the argument tells us that these are rotated $\pi/3$ radians clockwise. The coefficient on the cosine is $e = 1/2$ which also tells us that $p = 2$. This gives us everything we need to know to graph the ellipse.

Chapter 15

Modeling Mathematics

15.1 Cavalieri's Principle

According to **Cavalieri's principle**, if all parallel horizontal slices of an upright volume have the same area, then the volume is the area of a slice times the height.

For example, for a cylinder, the volume is the area of the base (or any horizontal slice) times the height of the cylinder. The same is true for cubes and prisms.

15.2 Cross-sections and Rotations

Be able to identify the cross-sections of three-dimensional objects and the three-dimensional objects formed by rotating a curve about a line.

Example 15.2.1

What is the volume of the three-dimensional object formed by rotating the curve $y = \sqrt{25 - x^2}$ about the x -axis?

The given equation describes the upper half of a circle with radius 5. If it is rotated about the x -axis, it forms a sphere of radius 5 and volume

$$V = \frac{4}{3}\pi r^3 = 524.$$

15.3 Tangents to a Circle

From a point outside of a circle, there are two lines that pass through the point P and are tangent to the circle. How do you calculate those lines and the points of tangency?

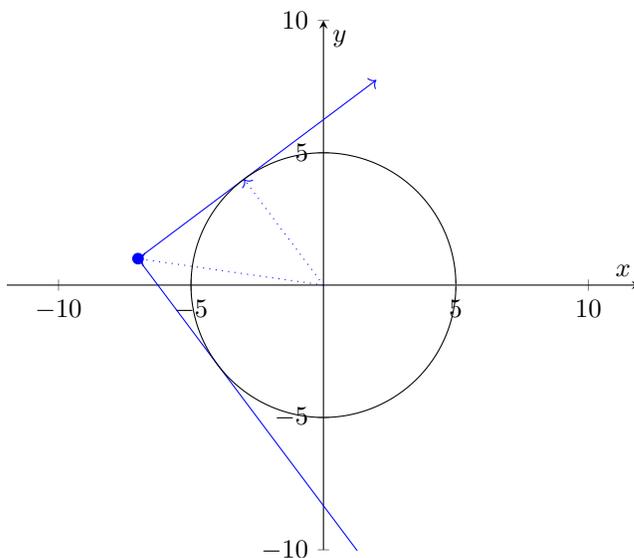
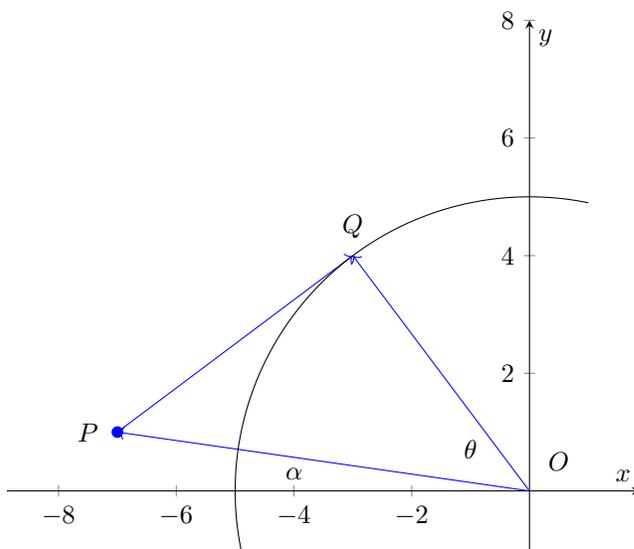
The first step is to draw a diagram. The key is to understand that at the point of tangency on the circle, the tangent line is at right angles to the line connecting the point of tangency to the center of the circle. This allows us to create a right triangle such that the tangent line is the long leg, the radius of the circle forms the short leg, and the line from P to the center of the circle forms the hypotenuse.

We know the radius of the circle, and therefore the length of the short side of the triangle. Using the distance formula, we can calculate the length of the hypotenuse. With these two, we can use the Pythagorean theorem to find the points of tangency on the circle.

The algebra becomes really messy if the circle is not at the origin.

If you know trigonometry, finding the points of tangency becomes a little less messy. If $O = (x_1, y_1)$ is the origin point, $P = (x_2, y_2)$ is the point through which both tangent points pass, and $Q = (x_3, y_3)$ is one of the points of tangency on the circle, then the length

of each side of the triangle OPQ can be found using the Pythagorean theorem as in the algebraic method. However, now we also consider the triangle formed by O , P , and the x -axis. We'll call the angle that OP makes with the x -axis, α , then $\alpha = \tan^{-1}(y_2/x_2)$. Next, we call the angle between OP and OQ , θ . We can easily find this angle since its tangent is the far leg of OPQ divided by the adjacent leg of OPQ . Notice that the adjacent leg of OPQ is simply r —the radius of the circle. Finally, the point of tangency Q is given by $x_3 = r \cos(\alpha + \theta)$ and $y_3 = r \sin(\alpha + \theta)$.



Example 15.3.1

Determine the tangent lines and tangent points for the two lines that go through the point $(-7, 1)$ and are tangent to the circle $x^2 + y^2 = 25$.

First, we diagram the situation. From the form of the equation of the circle, we know that the center lies at the origin. We also know that its radius is 5. In our diagram, the tangent lines each form the long legs of a right triangle with a shared hypotenuse. From the distance formula, we can calculate the the length of

the hypotenuse as

$$d = \sqrt{(-7)^2 + (1)^2} = \sqrt{50}.$$

We know that the short leg of each triangle is 5 units long since it is equal to the radius of the circle. From the Pythagorean theorem, we know that the length of the long leg squared, plus the length of the short leg squared equals the length of the hypotenuse squared, so for the upper point of tangency

$$\begin{aligned}(x + 7)^2 + (y - 1)^2 + 25 &= (\sqrt{50})^2 \\ x^2 + 14x + y^2 - 2y + 25 &= 0.\end{aligned}$$

Since the tangency point is on the circle, and the equation of the circle is $x^2 + y^2 = 25$, we can solve the equation of the circle for y to get $y = \pm\sqrt{25 - x^2}$, and substitute this into the equation above to get

$$x^2 + 14x + \left(\sqrt{25 - x^2}\right)^2 - 2\left(\sqrt{25 - x^2}\right) + 25 = 0,$$

which simplifies to $x^2 + 7x + 12 = 0$. Applying the quadratic formula gives us $x = -3, -4$. The $x = -3$ tangency point corresponds to the upper tangency point, so we plug this into the formula for the upper part of the circle to get $y = \sqrt{25 - (-3)^2} = 4$. The $x = -4$ tangency point corresponds to the lower tangency point, so we plug this into the formula for the lower part of the circle to get $y = -\sqrt{25 - (-4)^2} = -3$. This tells us that the points of tangency are $(-3, 4)$ and $(-4, -3)$. When it comes to finding the tangent lines, we have two points on each line and we can use this fact to find the slope of each tangent line. Then we can use the point-slope form of a line to find the tangent line.

For the upper tangent point, $m = (4 - 1)/(-3 - (-7)) = 3/4$ and so the line is

$$y - 4 = \frac{3}{4}(x + 3) \implies y = \frac{3}{4}x + \frac{25}{4}.$$

For the lower tangent point, $m = (-3 - 1)/(-4 - (-7)) = -4/3$ and so the line is

$$y + 3 = -\frac{4}{3}(x + 4) \implies y = -\frac{4}{3}x - \frac{25}{3}.$$

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