Linear Algebra
Class Notes

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Preface

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Updates

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Chapter 1

Introduction

A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b is a real or complex constant, the coefficients a_n are real or complex numbers, and x_n are the variables. The distinguishing feature of linear equations is that the variable is always to the first power. A 2-variable linear equation forms a straight line when graphed. The number of variables in the equation determines the dimension. For example, a linear equation of three variables forms a plane in a 3D graph.

A **system of linear equations** is a set of linear equations involving the same variables that is to be solved simultaneously.

The **solution** of a system of equations is the set of numbers that when substituted for the variables, makes the system of equations true. The **solution set** is the set of all possible solutions to the set of linear equations.

There are a number of ways to solve a system of linear equations (SLEs):

1. We can solve one equation for one variable, then plug this into the other equations, and simplify those in the same manner.
2. We can multiply each equation by constants and then add them so that one variable cancels.
3. We can use matrices.

There are three possible results with SLEs:

- There is no solution. Graphically, this is when the lines are parallel. Such a system is **inconsistent**. When solving such a system by adding, you will run into a contradiction (e.g. $0 = 2$).
- There is exactly one solution. Graphically, this is where the lines intersect. Such a system is **consistent**. When solving such a system by adding, you will get a single solution (e.g. $x = 1$).
- There are infinitely many solutions. Graphically, this is where the lines coincide (i.e. they are the same lines). Such a system is **consistent**. When solving such a system by adding, you will get infinite solutions (e.g. $0 = 0$).

Consider the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This is called a 2×2 matrix where the first number refers to the number of rows and the second number refers to the number of columns. The diagonal from a to d is called the **main diagonal** and the other diagonal is called the **secondary diagonal**.

Consider the SLE

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\ a_7x_1 + a_8x_2 + a_9x_3 &= b_3. \end{aligned}$$

The **coefficient matrix** is the matrix of the coefficients of the SLE

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}.$$

The **augmented matrix** is the coefficient matrix with an added column containing the constants.

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \\ a_7 & a_8 & a_9 & b_3 \end{array}\right]$$

The vertical line in the augmented matrix is not essential, but it helps to visually clarify that it is an augmented matrix.

Matrices can be used to solve SLEs systematically. An augmented matrix, is in fact, an SLE. Just like we can manipulate an SLE by adding and subtracting equations from one another and by multiplying equations by constants, we can do the same with the augmented matrix corresponding to the SLE. The allowed manipulations are called the **elementary row operations**.

- We can swap rows.
- We can multiply rows by a nonzero constant. This is called **scaling**.
- We can replace a row with itself plus another scaled row.
- We cannot do anything to the columns.

An important note on row operations is that they're all **reversible**. Whatever legal row operations that we performed on a matrix, we could undo them all to retrieve the original matrix.

We may denote row operations using notation of the form $R_1 = R_1 + 7R_3$, which means replace row 1 with row 1 plus 7 times row 3.

Two matrices are **row equivalent** if and only if elementary row operations can be used to transform one into the other. Row equivalent matrices have the same solution set.

When we see an SLE, we want to know

1. Is it consistent?
2. If so, do we have a unique solution. This is actually the easier of the two. We can answer these questions without fully solving the system. We only need to have the augmented matrix in the upper triangular form.

Upper triangular form or **row echelon form** is when there are only zeros under the main diagonal of the coefficient portion of the augmented matrix. For example, the matrix

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ 0 & a_5 & a_6 & b_2 \\ 0 & 0 & a_9 & b_3 \end{array}\right]$$

is in upper triangular form. For a matrix to be in echelon form

1. All nonzero rows must be above rows with nothing but zeros.
2. The leading entry (i.e. first nonzero number in the row) of each row must be in the column to the right of the leading entry of the row above it. That is, two leading entries cannot be on top of each other.
3. All entries in a column below a leading entry must be zero.

For a matrix to be in **reduced row echelon form**

1. It must be in echelon form.
2. The leading entries of each nonzero row must be 1.
3. All entries in a column *above* a leading entry must be zero.

The matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

is in reduced row echelon form. To check that a matrix is in reduced echelon form, it is not enough to note that the diagonals are 1 and the other values are zero. You need to refer to the definition of reduced echelon form. For example, the following matrix is in reduced echelon form because it fulfills all of the requirements

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

For a given matrix, there are multiple resulting row echelon matrices that one can transform it into depending on the row operation used. However, for any given matrix, there is one and only one reduced row echelon form. Every matrix can be put in echelon and reduced echelon forms.

In a matrix, a **pivot position** is a location that corresponds to a leading 1 in the same matrix when it is in reduced echelon form. In other words, you must put a matrix into echelon form to find the pivot positions. A **pivot column** is any column in the matrix containing a pivot position.

The algorithm for systematically turning any matrix into echelon form is

1. Note that the left-most nonzero column is a pivot column. The top entry in that column is a pivot position. A pivot position cannot be zero, so if the top left entry of the matrix is zero, swap the top row with another row so that the top-left element is nonzero.
2. Use row operations to convert all entries below the pivot position noted into zeros.
3. Now, ignore the first row and the first column and repeat 1–2 with the submatrix that remains, that is, until the elements below pivot positions are zero. Repeat this process until no more nonzero rows are left to modify. The matrix is now in echelon form. If you're working with an augmented matrix, you can now tell whether or not the system is consistent. It is inconsistent (i.e. has no solution) if the bottom row tells you something like $0 = 2$.
4. To get the matrix into reduced echelon form, you have to perform this final step. Scale all pivots to 1 (i.e. multiply the row by the reciprocal of the leading entry), and zero out all the entries in the column above each pivot. To do this, start with the rightmost pivot, scale it to one, and eliminate the entries above it. Repeat until your matrix is in reduced echelon form.

Tip

On exams, you may not be allowed a graphing calculator since those are able to perform a lot of the matrix computations learned in this course. Doing a lot of arithmetic (when doing row operations) by hand, may slow you down. See if it's possible to use a non-scientific, non-graphing calculator on the exams.

The variables that correspond to pivot positions are called **basic variables**. The rest are **free variables**. This is just by convention. Consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Since the coefficient portion of this matrix has five columns, we know that the corresponding system of equations has five variables, which might by x_1, x_2, x_3, x_4, x_5. The first, third, and fifth column contains pivot positions, so x_1, x_3, and x_5 are basic variables and x_2 and x_4 are free variables. This means x_1, x_3, and x_5 are defined in terms of x_2, and x_4, and x_2 and x_4 can take on any values whatsoever. Writing the solution set for an SLE containing only basic variables is a simple matter of listing each variable and its value. For an SLE containing free variables, write the solution in terms of the free variables. For example, the augmented matrix shown above corresponds to the SLE

$$\begin{aligned} x_1 + 5x_2 + x_4 &= 2 \\ x_3 + 2x_4 &= 4 \\ x_5 &= 6 \end{aligned}$$

so the solution set would be written as

$$\begin{cases} x_1 &= 2 - 5x_2 - x_4 \\ x_3 &= 4 - 2x_4 \\ x_5 &= 6 \end{cases}$$

This type of solution set is known as **parametric form** since the free variables are acting as parameters of the solution set.

If the right-most column in an augmented matrix (i.e. the column containing the constants) is a pivot column then there are no solutions–the SLE is inconsistent. For example, if the augmented matrix claims that $0 = 3$, it is inconsistent.

If it is not a pivot column, then it is consistent. If there are free variables, there are infinite solutions. If there are no free variables, there is a unique solution.

1.1 Vectors and Vector Equations

A one-column matrix can be referred to as a **column vector** or even just a **vector**. Similarly, vectors can be represented by one-column matrices.

$$\vec{x} = \langle x_1, x_2 \rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Addition and scalar multiplication works just like we're used to with vectors:

$$c\vec{x} = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}.$$

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}.$$

Notice that the number of dimensions the vector is in is the same as the number of rows in the column vector. For a vector in $\mathbb{R}^n$

$$\vec{\boldsymbol{x}} = \langle x_1, x_2, \ldots x_n \rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r$, are vectors in $\mathbb{R}^n$, and $c_1, c_2, \ldots c_r$ are arbitrary scalars, then the vector

$$\vec{\boldsymbol{u}} = c_1\vec{\boldsymbol{v}}_1 + c_2\vec{\boldsymbol{v}}_2 + \cdots + c_r\vec{\boldsymbol{v}}_r,$$

is a **linear combination** of $\vec{\boldsymbol{v}}_1$ to $\vec{\boldsymbol{v}}_r$ with weights c_1 to c_r. In simple terms, any vector that can be created by adding multiples of some set of vectors is a linear combination of that set of vectors. For example, $\vec{\boldsymbol{u}} = 3\vec{\boldsymbol{v}}_2$ is a linear combination of $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r$. All the other coefficients are just 0.

To understand the concept of linear combinations, you could think of things other than vectors. For example, the number 16 is a linear combination of $\{2, 3\}$ because $4(3) + 2(2) = 16$.

Example 1.1.1

Is the vector $\vec{\boldsymbol{u}}$ a linear combination of $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ if

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 39 \\ 54 \\ 69 \end{bmatrix}, \quad \vec{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

What it's asking is if

$$\vec{\boldsymbol{u}} = c_1\vec{\boldsymbol{v}}_1 + c_2\vec{\boldsymbol{v}}_1,$$

so we need to see if there are scalar values of c_1 and c_2, that make the equation true. Since the vectors are column vectors, what we have is an SLE that can be represented by the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & 39 \\ 2 & 5 & 54 \\ 3 & 6 & 69 \end{array}\right]$$

The reduced echelon form of this augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

which tells us that it is a linear combination if $c_1 = 7$ and $c_2 = 8$.

The form of the **general vector equation** is

$$x_1\vec{\boldsymbol{a}}_1 + x_2\vec{\boldsymbol{a}}_2 + \cdots + x_n\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{b}}.$$

The **span** of a set of vectors $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r\}$ is the set of all vectors that are linear combinations of the vectors in $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r\}$. Since there are infinite real numbers that can be used as scalar multipliers, the number of vectors in $\text{span}\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r\}$ is infinite.

Example 1.1.2

Is $\vec{\boldsymbol{0}}$ in the span of $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots \vec{\boldsymbol{v}}_r\}$?
Yes, because

$$\vec{\boldsymbol{0}} = 0\vec{\boldsymbol{v}}_1 + 0\vec{\boldsymbol{v}}_2 + \cdots + 0\vec{\boldsymbol{v}}_r.$$

In $\mathbb{R}^3$, $\text{span}\{\vec{\boldsymbol{v}}\}$ is a straight line. This is because the linear combinations of a single vector are all in the same direction–they're just all the possible scaled versions of that vector. In $\mathbb{R}^3$, $\text{span}\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2\}$ is a plane provided that the two vectors are not pointing in the same direction. In $\mathbb{R}^3$, $\text{span}\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3\}$ is $\mathbb{R}^3$ provided that $\vec{\boldsymbol{v}}_3$ is not in $\text{span}\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2\}$ and $\vec{\boldsymbol{v}}_2$ is not in $\text{span}\{\vec{\boldsymbol{v}}_1\}$.

When asked to determine if a given vector is in the span of other given vectors, just form a vector equation, and solve the corresponding SLE as done in the example above.

Each column in a matrix can be thought of as a vector, so a matrix can also be written in the form of a single row matrix containing vector elements

$$A = \begin{bmatrix} \vec{\boldsymbol{a}}_1 & \vec{\boldsymbol{a}}_2 & \cdots & \vec{\boldsymbol{a}}_n \end{bmatrix}$$

Since a single column matrix is a vector, we can write the product of a matrix and a vector as

$$\begin{aligned} A\vec{\boldsymbol{x}} &= \begin{bmatrix} \vec{\boldsymbol{a}}_1 & \vec{\boldsymbol{a}}_2 & \cdots & \vec{\boldsymbol{a}}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \vec{\boldsymbol{a}}_1 x_1 + \vec{\boldsymbol{a}}_2 x_2 + \cdots + \vec{\boldsymbol{a}}_n x_n. \end{aligned}$$

1.2 The Matrix Equation

In general, the **matrix vector equation** is

$$A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}.$$

Notice that $\vec{\boldsymbol{x}}$ must have the same number of rows as A has columns in order for them to be multiplied like this. In other words, if A is an $m \times n$ matrix, then $\vec{\boldsymbol{x}}$ must be in $\mathbb{R}^n$. Even if A has multiple rows, the result of $A\vec{\boldsymbol{x}}$ will be a one-column matrix, that is, the vector $\vec{\boldsymbol{b}}$.

Recall that to multiply two matrices, you multiply the elements in the rows of the first matrix with the columns of the second matrix and add them. Essentially, you take the dot product of a row in the first matrix with the corresponding column in the second matrix. Notice that you only care about the rows in the first matrix and the columns in the second matrix and your product will be a matrix with the same number of rows as the first matrix and the same number of columns as the second matrix.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \end{bmatrix} \times \begin{bmatrix} C_1 & C_2 & C_3 & \cdots \end{bmatrix} = \begin{bmatrix} R_1 \bullet C_1 & R_1 \bullet C_2 & R_1 \bullet C_3 & \\ R_2 \bullet C_1 & R_2 \bullet C_2 & R_2 \bullet C_3 & \cdots \\ R_3 \bullet C_1 & R_3 \bullet C_2 & R_3 \bullet C_3 & \\ & \vdots & & \end{bmatrix}$$

Example 1.2.1

Write the vector expression

$$2\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 - 3\vec{\boldsymbol{v}}_3,$$

as a matrix vector product.

$$\begin{bmatrix} \vec{\boldsymbol{v}}_1 & \vec{\boldsymbol{v}}_2 & \vec{\boldsymbol{v}}_3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 2\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 - 3\vec{\boldsymbol{v}}_3$$

Example 1.2.2

Write the SLE

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ 2x_2 - 3x_3 &= 4 \end{aligned}$$

as a matrix equation.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

We can think of SLEs as augmented matrices, as vector equations, or as shown in the example above, as matrix equations.

If A is an $m \times n$ matrix, then the following four statements are all equivalent

- For each $\vec{\boldsymbol{b}}$ in $\mathbb{R}^m$, $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a solution.
- Each $\vec{\boldsymbol{b}}$ in $\mathbb{R}^m$, is a linear combination of columns in A.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot position in every row.

Although the equivalent statements above may not be true, it is helpful to know that if any one of them is true then the other three are true as well. If any one of them is false, then the other three are false as well.

A system of linear equations is **homogeneous** if it can be written in the form

$$A\vec{x} = \vec{0}.$$

That is, if all the constants of an SLE are zero, then it is homogeneous. A homogeneous SLE, necessarily has at least one solution–the one where $\vec{x} = 0$ (i.e. all the elements in the column vector are zero), called the **trivial solution**. A **non-trivial solution** exists when a non-zero vector $\vec{x}$ satisfies the matrix equation $A\vec{x} = \vec{0}$.

A non-trivial solution exists if and only if the SLE has at least one free variable. This follows from the fact that the rightmost column in an augmented matrix cannot be a pivot column if the SLE is to be consistent. Consider the augmented matrix of a homogeneous SLE. Since the far-right column contains only zeros (this is unchanged by elementary row operations), then in RREF, the column just to the left of the far-right, cannot be a pivot column in order for the SLE to be consistent. If it is a pivot column, it's asserting that $1x_3 = 0$, for example, making the SLE is inconsistent.

When writing the solution set for an SLE, write it as a vector or sum of vectors each multiplied by the corresponding free variable. For example, in the one line SLE $x_1 + 2x_2 + 3x_3 = 0$, the solution is $x_1 = -2x_2 - 3x_3$ where x_2 and x_3 are free variables. The solution set can be written as

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Notice that this could also be written as

$$\vec{x} = x_2\vec{u} + x_3\vec{v}, \text{ where } \vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

If the non-homogeneous equation $A\vec{x} = \vec{b}$ is consistent for $\vec{b}$ and $\vec{p}$ is a solution, then the solution set of $A\vec{x} = \vec{b}$ is the set of vectors $\vec{w} = \vec{p} + \vec{v}h$ where $\vec{v}h$ is any solution to the homogeneous equation $A\vec{x} = \vec{0}$. In other words, the only difference in the solution set of the non-homogeneous case is an added vector of constants.

For example, if $A\vec{x} = \vec{0}$ has one free variable, the solution set is a line running through through the origin. It must necessarily go through the origin since it contains the trivial solution $\vec{x} = 0$. Then the solution set $\vec{x} = \vec{p}$ to a consistent equation $A\vec{x} = \vec{b}$ will be a translation of the earlier line. In other words, it will take every point on the first line and translate it by $+\vec{p}$ to form a new line that does not go through the origin. Similarly, if the homogeneous solution set is a plane, the non-homogeneous solution set will be a translated version of that plane.

1.3 Linear Independence

A set of vectors $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \dots, \vec{\boldsymbol{v}}_p\}$ is **linearly independent** if and only if

$$x_1\vec{\boldsymbol{v}}_1 + x_2\vec{\boldsymbol{v}}_2 + \cdots + x_p\vec{\boldsymbol{v}}_p = \vec{\mathbf{0}},$$

where x_i are arbitrary constants, has only the trivial solution where $x_1 = x_2 = \cdots = x_p = 0$. Otherwise, if there are non-trivial solutions, the set of vectors is **linearly dependent**. You can think of the equation $x_1\vec{\boldsymbol{v}}_1 + x_2\vec{\boldsymbol{v}}_2 + \cdots + x_p\vec{\boldsymbol{v}}_p = \vec{\mathbf{0}}$ as the vector equation $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$, with A being the matrix $[\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \dots, \vec{\boldsymbol{v}}_p]$.

Recall that the homogeneous equation $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ has non-trivial solutions if and only if there are free variables. In other words, if the matrix equation has free variables then it has infinitely many solutions, which implies that there is more than just the trivial solution, and so the set is linearly dependent. If the SLE is consistent, but there are no free variable, then there is a unique solution. If a homogeneous equation has a unique solution then it must be the trivial solution since the trivial solution is always a solution.

To check if a set of vectors or the columns of a matrix are linearly independent, you set up an augmented matrix where the constants are zero. If there is a unique solution, they are linearly independent, and if there are infinite solutions, they are linearly dependent.

Think of linear independence in terms of vectors. If you have two vectors that are not pointing in the same direction, they will span $\mathbb{R}^2$. If you add a third vector that is in the same plane, and therefore can be represented as a linear combination of the other two vectors, the set is then linearly dependent. If, on the other hand, the third vector is not in the same plane, then the set of three is linearly independent since none of the vectors can be written as linear combinations of the rest.

Example 1.3.1

Are the following vectors linearly independent?

$$\vec{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{\boldsymbol{v}}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix},$$

To check if the vectors are linearly independent, use the augmented matrix for the homogeneous matrix equation and see if there are non-trivial solutions.

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is a free variable, so there are infinite solutions. This implies that the trivial solution is not the only one, so the vectors are linearly dependent.

A single vector $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}$ because there are infinite solutions for $x\vec{v} = \vec{0}$. Otherwise, if $\vec{v} \neq \vec{0}$, then there is a unique solution $x = 0$ and the vector is linearly independent.

Two vectors $\{\vec{v}_1, \vec{v}_2\}$ are linearly dependent if and only if one of them is proportional to the other, that is $\vec{v}_1 = c\vec{v}_2$ for some arbitrary constant c. For example, the pair of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix},$$

are linearly dependent because $\vec{v}_2 = 3\vec{v}_1$. To check if a pair of vectors are linearly independent just check if one is a multiple of the other.

For more than two vectors, use the augmented matrix method used in the example earlier.

Theorem: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly dependent if there is some $1 \leq k \leq p$ such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{v}_k$. This is just saying that a set of vectors is linearly dependent if any one of them can be written as a linear combination of any of the others. In other words, if a subset of the vectors is linearly dependent then the whole set is linearly dependent. This can actually be written in a stricter form:

Theorem: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly dependent if and only if there is some $1 \leq k \leq p$ such that $c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k$. In other words, if you reach any vector in the set that can be written as a linear combination of the preceding ones then the set is linearly dependent. This implies that if you have a set of linearly independent vectors, adding a single vector that is a linear combination of the ones in the set makes the set linearly dependent.

Theorem: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ containing the zero vector is linearly dependent. This is because the zero vector multiplied by any arbitrary constant is a non-trivial solution to the matrix equation. For example, the set $\{\vec{v}_1, \vec{0}, \vec{v}_3\}$ is linearly dependent because

$$\vec{x} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$$

is a non-trivial solution to the homogeneous matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{0} & \vec{v}_2 \end{bmatrix} \vec{x} = \vec{0}.$$

Theorem: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$. In other words, if the number of vectors is larger than the dimension of the vectors, then the set is linearly dependent. Think of the $\mathbb{R}^2$ example. If you have three vectors in $\mathbb{R}^2$, then they must necessarily be linearly dependent because you can't have three linearly independent vectors in a plane since it only takes two of them to span the entire plane.

1.4 Linear Transformations

A **linear transformation** takes in a vector in $\mathbb{R}^n$ and returns a vector in $\mathbb{R}^m$. Notice that the dimensions don't have to be the same. In calculus courses, transformations are called vector fields.

A transformation may be denoted

$$T(\vec{v}) = \vec{u},$$

and means that the transformation T, transforms vectors $\vec{v}$ into vectors $\vec{u}$. To denote the dimensions of the vectors, we write

$$T : \mathbb{R}^n \to \mathbb{R}^m,$$

meaning that T is a transformation that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. The dimension of the input vectors gives the **domain** of the transformation, the dimension of the output vectors gives the **codomain**, and the set of possible output vectors is the **range** of the transformation. A specific output vector $T(\vec{x})$ is called the **image** of $\vec{x}$.

For example, the transformation $T(\vec{v}) = \vec{u}$, where $T : \mathbb{R}^2 \to \mathbb{R}^3$, maps any 2-dimensional input vector $\vec{v}$ to a three dimensional vector $\vec{u}$. The input is a plane (all of $\mathbb{R}^2$), and the output might be a plane in $\mathbb{R}^3$. in that case, the domain of T is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$, and the range is the specific plane in $\mathbb{R}^3$ that the output vectors span.

We can think of linear transformations as matrices and vice versa, so we can write transformations in the form

$$T(\vec{x}) = A\vec{x},$$

which shows that applying the transformation T to a vector $\vec{x}$ is equivalent to multiplying $\vec{x}$ by some matrix A. If A is an $m \times n$ matrix, then the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. This is because to be able to multiple a vector by a matrix with m rows and n columns, the vector must have n rows, that is, it must be in $\mathbb{R}^n$. Second, since the matrix has m rows, the product will also have m rows, so the resulting vector will be in $\mathbb{R}^m$.

Example 1.4.1

We have the transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 17 \\ 39 \\ 61 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -3 \\ 12 \\ 6 \end{bmatrix}.$$

Find the image of $\vec{u}$, find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{v}$, and see if $\vec{w}$ is in the range of T.

The image of $\vec{u}$ is just $T(\vec{u})$, so we do the matrix vector multiplication to get

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}.$$

Notice that the domain of T is $\mathbb{R}^2$ since the input vectors are two dimensional. The codomain of T is $\mathbb{R}^3$ since the output vectors are three dimensional.

To find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{v}$, we have to solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \\ 61 \end{bmatrix}$$

for $\vec{x}$. Remember that we can treat this as a system of equations. We start by putting it in augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & 17 \\ 3 & 4 & 39 \\ 5 & 6 & 61 \end{array}\right],$$

for which the RREF is

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array}\right],$$

so the solution is $\vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. To figure out if $\vec{w}$ is in the range of T, we need to know if there is any possible $\vec{x}$ for which $T(\vec{x}) = \vec{w}$. Again, we put it in an augmented matrix and use row operations to get the RREF. In this case, we find that the system is inconsistent, so $\vec{w}$ is not in the range of T.

A transformation T is linear if and only if $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$. Another way to think of it is to combine the two rules and say that if T is linear, then

$$\begin{aligned} T(\vec{0}) &= \vec{0}, \text{ and} \\ T(c\vec{u} + d\vec{v}) &= cT(\vec{u}) + dT(\vec{v}). \end{aligned}$$

One corollary of this is that a linear transformation T satisfies the **superposition principle**

$$T\left(c_1\vec{v_1} + \cdots + c_n\vec{v}_n\right) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n).$$

To prove that if T is linear then $T(\vec{0}) = \vec{0}$, we start by assuming that T is linear and that $T(\vec{0}) = \vec{v}$ where $\vec{v} \neq \vec{0}$. Then we note that $T(c\vec{0}) = T(\vec{0}) = \vec{v}$, but also, $T(c\vec{0}) = cT(\vec{0}) = c\vec{v}$. Since $c\vec{v} \neq \vec{v}$, we have a contradiction.

Recall that transformations are matrices, so matrix transformations are linear:

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v} \\ A(c\vec{v}) &= cA\vec{v}. \end{aligned}$$

Example 1.4.2

Show that $T(\vec{x}) = 2\vec{x}$ is a linear transformation.
To show that T is linear, we have to show that $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.

$$\begin{aligned} T(c\vec{u} + d\vec{v}) &= 2(c\vec{u} + d\vec{v}) \\ &= 2c\vec{u} + 2d\vec{v}) \\ &= c2\vec{u} + d2\vec{v} \\ &= cT(\vec{u}) + dT(\vec{v}). \end{aligned}$$

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then there is a unique $m \times n$ **standard matrix** A of the form

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}.$$

such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$. The vector $\vec{e}_i$ is the ith column of the identity matrix I_n. These vectors are called the **standard basis vectors**. In other words, if you know what the transformation T does to the columns of the identity matrix, you can construct the standard matrix A that characterizes the transformation.

Example 1.4.3

Find the standard matrix A for the transformation $T(\vec{x}) = 2\vec{x}$ given that $\vec{x} \in \mathbb{R}^3$.

To find the standard matrix, we need to find $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$ since our domain is $\mathbb{R}^3$. The standard basis vectors $\vec{e}_i$ are just the columns of the identity matrix I_3, so

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It's easy to see what T does to each of these vectors since it only multiplies every entry by 2, so

$$T(\vec{e})_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

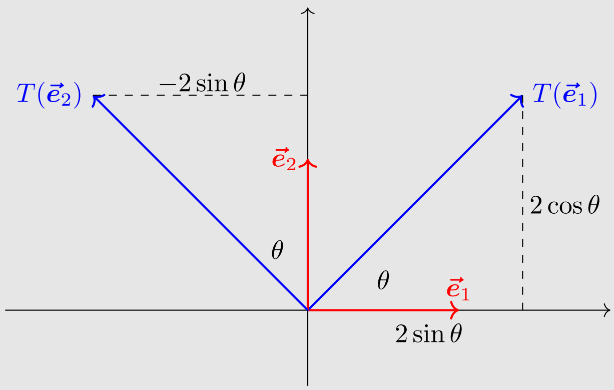
Putting the three results together, we have that our standard matrix is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 1.4.4

Find the standard matrix for a transformation that rotates two dimensional vectors counterclockwise by angle θ and doubles their length.

To find the standard matrix, we need to find $T(\vec{e})_1$ and $T(\vec{e})_2$. In other words, we need to understand how T transforms the standard basis vectors. To do this, it helps to draw a graph in this case.



Notice that $\vec{e}_1$ transforms from $\langle 1,0\rangle$ to $\langle 2\cos\theta, 2\sin\theta\rangle$ and $\vec{e}_2$ transforms from $\langle 0,1\rangle$ to $\langle -2\sin\theta, 2\cos\theta\rangle$. This tells us that

$$T(\vec{e}_1) = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} -2\sin\theta \\ 2\cos\theta \end{bmatrix}.$$

So from the theorem given above, the standard matrix for this transformation is

$$A = \begin{bmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{bmatrix}.$$

A function f from a set X to a set Y is **onto** or **surjective** if for every y in Y, there is an x in X such that $f(x) = y$. In other words, a function $y = f(x)$ is onto if you can get any y-value as output provided that you choose the input x appropriately. The function $y = x$ is onto, for example, but $y = x^2$ is not because you can't get the negative numbers as output.

The concept of onto functions also applies to transformations. For a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ then T is onto if and only if for all vectors $\vec{y}$ in $\mathbb{R}^m$, there is at least one $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

A function f from a set X to a set Y is **one-to-one** or **injective** if for all y in Y, there is at most one x in X such that $f(x) = y$. In other words, a function $y = f(x)$ is one-to-one only if multiple x-values can never return the same y-value. One-to-one functions pass the horizontal line test.

The concept of one-to-one functions also applies to transformations. For a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ then T is one-to-one if there is at most one (could be 0) $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

A transformation $T(\vec{x}) = \vec{y}$ is both onto and one-to-one, that is, it is **bijective**, if there is exactly one $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

Example 1.4.5

Determine if $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is an onto transformation if

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To answer this, we need to know if $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$ has a solution for every $\vec{\boldsymbol{y}}$. Think of forming the augmented matrix with a column for $\vec{\boldsymbol{y}}$ on the right side. Since the last column of A (which is in echelon form) is a pivot column, we know that there will always be a solution. Because the bottom right entry is nonzero, we know that the SLE is not inconsistent. So a transformation is onto if the last column in the standard matrix is a pivot column.

Theorem: $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{0}}$ has only the trivial solution.

Example 1.4.6

Determine if $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is a one-to-one transformation if

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To answer this, we choose $\vec{\boldsymbol{y}} = \vec{\boldsymbol{0}}$, so that $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$. Since there is a free variable, there are infinite vectors $\vec{\boldsymbol{x}}$ that satisfy $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$, so T is not a one-to-one transformation. To check if a transformation is one-to-one, we just have to see if there are any free variables.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation where $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$, then

1. T is onto if and only if the columns of A span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of A are linearly independent. If the columns are linearly independent, there won't be a free variable.

This theorem allows us to classify a lot of transformations as onto or not or one-to-one or not just by looking at their domains and codomains. For example, a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ must have a standard matrix that is 3×2. Since A has only two columns and since two columns cannot possibly span $\mathbb{R}^3$, we know that T is not onto. The columns may or may not be independent, so T may or may not be one-to-one. Similarly, a transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ must have a standard matrix that is 2×3. Since A has three columns, they may or may not span $\mathbb{R}^3$ so T may or may not be onto. Since there are more columns than rows, A must contain a free variable, so it cannot possibly be a one-to-one transformation.

1.5 Applications

Example 1.5.1

Balance the chemical equation

$$C_2H_6 + O_2 \longrightarrow CO_2 + H_2O$$

using linear algebra.

To balance the chemical equation, we need to find values for the coefficients such that the number of atoms of each element is the same on both sides of the equation.

$$x_1C_2H_6 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O$$

In other words, we want

$$x_1\begin{bmatrix} C \\ H \\ O \end{bmatrix} + x_2\begin{bmatrix} C \\ H \\ O \end{bmatrix} = x_3\begin{bmatrix} C \\ H \\ O \end{bmatrix} + x_4\begin{bmatrix} C \\ H \\ O \end{bmatrix}$$

Rearranging and filling in the values, we want that

$$x_1\begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting this into augmented matrix form gives us

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{7}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$$

which tells us that $x_1 = \frac{1}{3}x_4$, $x_2 = \frac{7}{6}x_4$, $x_3 = \frac{2}{3}x_4$, and x_4 is a free variable, so

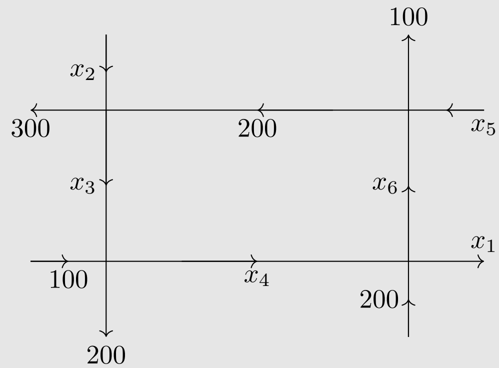
$$\frac{1}{3}x_4C_2H_6 + \frac{7}{6}x_4O_2 \longrightarrow \frac{2}{3}x_4CO_2 + x_4H_2O.$$

By convention, a balanced chemical equation is usually given without fractions (since partial atoms doesn't make a lot of sense), and in lowest form. So if we let $x_4 = 1$ and multiply across by 2 to eliminate the fraction, we get

$$2C_2H_6 + 7O_2 \longrightarrow 4CO_2 + 6H_2O.$$

Example 1.5.2

Find the missing values in the **network** using linear algebra.



With a network, it is assumed that the total stuff that is flowing is conserved at each node of the network. That is, for any node, the total incoming equals the total outgoing. This gives us a linear equation for each of the nodes

$$\begin{aligned} x_2 - x_3 &= 100 \\ x_5 + x_6 &= 300 \\ x_1 - x_4 + x_6 &= 200 \\ x_3 - x_4 &= 100. \end{aligned}$$

Furthermore, we can assume that the total flowing into the network equals the total flowing out of the network, which gives us the fifth linear equation

$$-x_1 + x_2 + x_5 = 300.$$

Putting the SLE into augmented matrix form gives us

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 1 & 300 \\ 1 & 0 & 0 & -1 & 0 & 1 & 200 \\ 0 & 0 & 1 & -1 & 0 & 0 & 100 \\ -1 & 1 & 0 & 0 & 1 & 0 & 300 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 0 & 1 & 200 \\ 0 & 1 & 0 & -1 & 0 & 0 & 200 \\ 0 & 0 & 1 & -1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the flow in the network is $x_1 = x_4 - x_6 + 200$, $x_2 = x_4 + 200$, $x_3 = x_4 + 100$, $x_5 = -x_6 + 300$, and x_4 and x_6 are free.

Example 1.5.3

In economics, an economy may be simply modeled by dividing it into sectors and quantifying the the input and output of each sector. If the value of each sector's total output is the "price" of that output, and if any sector's output is distributed in some way amongst all the sectors, then there exists an **equlibrium price** for each sector such that the income and expenses of each sector balances.

Suppose a small economy is divided into the three sectors oil, steel, and food. Suppose that oil sells 60% of its output to steel, 30 % to food, and keeps the rest. Steel sells 50% of its output to oil and 50% to food. Food sells 30% of its output to steel, 30% to oil, and keeps the rest. Find the equilibrium prices.

If the total output of each sector is designated p (with indices corresponding to the sector), then since oil purchases 0.1 of all oil, 0.5 of all steel output, and 0.3 of all food output, in order for income and expenses to balance, it must be the case that

$$p_O = 0.1p_O + 0.5p_S + 0.3p_F.$$

Similarly,

$$p_S = 0.6p_O + 0p_S + 0.3p_F,$$

and

$$p_F = 0.3p_O + 0.5p_S + 0.4p_F.$$

Simplifying the equations gives us an SLE with the augmented matrix

$$\left[\begin{array}{ccc|c} 0.9 & -0.5 & -0.3 & 0 \\ -0.6 & 1 & -0.3 & 0 \\ -0.3 & -0.5 & 0.6 & 0 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.75 & 0 \\ 0 & 1 & -0.75 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us that $p_O = p_S = 0.75p_F$. This tells us that the equilibrium prices are controlled by the price of food.

A difference equation or recurrence relation occurs when an initial value is known as well as a way to compute the next value. All values of interest can then be computed by repeated application of the difference equation. The same concept can be applied to matrices. For example, the equation

$$\vec{\boldsymbol{x}}_{n+1} = A\vec{\boldsymbol{x}}_n, \quad \text{where } n = 0, 1, 2 \ldots,$$

is called a **linear difference equation**. If we have a known vector $\vec{\boldsymbol{x}}_k$ and a matrix A to compute the next value, then $\vec{\boldsymbol{x}}_1 = A\vec{\boldsymbol{x}}_0$.

Example 1.5.4

Use a linear difference equation to calculate the population of a city and its suburbs in 2017 if the population of the city is 500,000 in 2015 and the population of the suburbs is 300,000. It is known that in any given year, the probability that a person in the city moves to the suburbs is 5% and the probability that a person moves from the suburbs to the city is 3%.

If 5% of the people in the city move to the suburbs in any given year, then 95% of them don't move. Second, if 3% of the suburban population moves to the city then 97% of them stay. We can represent this information with the matrix

$$A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}.$$

We can represent the populations in 2015 with the vector

$$\vec{\boldsymbol{x}}_0 = \begin{bmatrix} 500,000 \\ 300,000 \end{bmatrix},$$

then the population in 2016 is $\vec{\boldsymbol{x}}_1 = A\vec{\boldsymbol{x}}_0$ or

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 500,000 \\ 300,000 \end{bmatrix} = \begin{bmatrix} 484,000 \\ 316,000 \end{bmatrix},$$

and the population 2017 is $\vec{\boldsymbol{x}}_2 = A\vec{\boldsymbol{x}}_1$ or

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 484,000 \\ 316,000 \end{bmatrix} = \begin{bmatrix} 469,280 \\ 330,720 \end{bmatrix}.$$

So the city population in 2017 is 468,280 and the suburban population is 330,720.

1.6 Summary: Introduction

The following questions all ask the same thing.

1. $A\vec{x} = \vec{b}$, solve for x_1, x_2, and x_3
2. $A\vec{x} = \vec{b}$, solve for $\vec{x}$
3. Given a system of equations, solve for x_1, x_2, and x_3
4. Given the augmented matrix, row reduce it into reduced echelon form
5. Write $\vec{b}$ as a linear combination of the columns of A

They are all solved by creating an augmented matrix and row reducing it.

For a matrix to be in **echelon form**

1. All nonzero rows must be above rows with nothing but zeros.
2. The leading entry (i.e. first nonzero number in the row) of each row must be in the column to the right of the leading entry of the row above it. That is, two leading entries cannot be on top of each other.
3. All entries in a column below a leading entry must be zero.

For a matrix to be in **reduced row echelon form**

1. It must be in echelon form.
2. The leading entries of each nonzero row must be 1.
3. All entries in a column *above* a leading entry must be zero.

What can we tell about the solution set given an augmented matrix in echelon form?

1. Is the last column a pivot column?
   a) Yes: There is no solution
   b) No: Are there any free variables?
      i. Yes: There are infinite solutions
      ii. No: There is a unique solution

To say that $\vec{b}$ is in span$\{\vec{v}_1, \ldots, \vec{v}_p\}$ is the same as saying that $\vec{b}$ is a linear combination of $\{\vec{v}_1, \ldots, \vec{v}_p\}$. To find any $\vec{b}$ in some given span, just multiply the vectors in the set by arbitrary constants and add them up.

A system of linear equations is **homogeneous** if it can be written in the form

$$A\vec{x} = \vec{0}.$$

A homogeneous SLE, necessarily has the **trivial solution** $\vec{x} = 0$ A **non-trivial solution** exists when any non-zero vector $\vec{x}$ satisfies the matrix equation $A\vec{x} = \vec{0}$. A non-trivial solution exists if and only if the SLE has at least one free variable. This follows from the fact that the column just left of the rightmost column in an augmented matrix cannot be a pivot column if the SLE is to be consistent.

The solution of a consistent equation $A\vec{x} = \vec{b}$ with infinite solutions (i.e. it has a free variable) is a translated version of the solution set of $A\vec{x} = \vec{0}$. The general solution of $A\vec{x} = \vec{b}$ can be formed by adding the general solution of $A\vec{x} = \vec{0}$ and any particular solution of $A\vec{x} = \vec{b}$ (such as the particular solution that occurs when the free variables are zero).

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is **linearly independent** if and only if

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0},$$

where x_i are arbitrary constants, has only the trivial solution where $x_1 = x_2 = \cdots = x_p = 0$. Otherwise, if there are non-trivial solutions, the set of vectors is **linearly dependent**.

If the matrix equation $A\vec{x} = \vec{0}$ has free variables then it has infinitely many solutions, which implies that there is more than just the trivial solution, and so the set is linearly dependent. If the SLE is consistent, but there are no free variable, then there is a unique solution. If a homogeneous equation has a unique solution then it must be the trivial solution since the trivial solution is always a solution.

To check if a set of vectors or the columns of a matrix are linearly independent, you set up an augmented matrix where the constants are zero. If there is a unique solution, they are linearly independent, and if there are infinite solutions, they are linearly dependent.

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$. In other words, if the number of vectors is larger than the dimension of the vectors, then the set is linearly dependent.

A transformation T is linear if and only if $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$. Another way to think of it is to combine the two rules and say that if T is linear, then

$$\begin{aligned} T(\vec{0}) &= \vec{0}, \text{ and} \\ T(c\vec{u} + d\vec{v}) &= cT(\vec{u}) + dT(\vec{v}). \end{aligned}$$

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the standard matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix},$$

where $\vec{e}_i$ is the ith column of I_m.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if and only if for all vectors $\vec{y}$ in $\mathbb{R}^m$, there is at least one $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$. A transformation is onto if every *row* in A has a pivot position. That is, if the columns of A span $\mathbb{R}^m$.

The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if there is at most one (could be 0) $\vec{\boldsymbol{x}}$ in $\mathbb{R}^n$ such that $T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{y}}$. A transformation is one-to-one if A has no free variables.

A transformation $T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{y}}$ is both onto and one-to-one, if there is exactly one $\vec{\boldsymbol{x}}$ in $\mathbb{R}^n$ such that $T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{y}}$.

For an $m \times n$ matrix A, if and only if there is a pivot in every *column*, then

- the columns of A are linearly independent, and
- $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is one-to-one.

For an $m \times n$ matrix A, if and only if there is a pivot in every *row*, then

- the columns of A span $\mathbb{R}^m$, and
- $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is onto $\mathbb{R}^m$.

For an $n \times n$ matrix, if there is a pivot in every column then there is a pivot in every row.

Chapter 2

Matrix Algebra

If A is an $m \times n$ matrix, we can write the elements of A as a_{ij} where i is the row of the entry and j is the column of the entry.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

If $i = j$, then a_{ij} is a **diagonal entry**. All diagonal entries form the **main diagonal** of A. A **diagonal matrix** is a square matrix in which the nondiagonal entries are 0. A **zero matrix** is one in which all elements are zero.

Two matrices A and B are equal, $A = B$, if they are the same size and have the same entries.

2.1 Matrix Addition

Two matrices that are the same size can be added simply by adding their corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$$

2.2 Matrix Scalar Multiplication

Matrices can be multiplied by scalars simply by multiplying each entry of the matrix by the scalar.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

2.3 Matrix Multiplication

Matrix multiplication can be thought of as a series of transformations. For example, if $\vec{x}$ is a vector in $\mathbb{R}^p$, then multiplying it by the $n \times p$ matrix B is equivalent to applying the transformation $T : \mathbb{R}^p \to \mathbb{R}^n$, where $T(\vec{\boldsymbol{x}}) = B\vec{\boldsymbol{x}}$. Notice that $B\vec{\boldsymbol{x}}$ is a new vector in $\mathbb{R}^n$. If we multiply this by the $m \times n$ matrix A, we are making the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, where $T(B\vec{\boldsymbol{x}}) = A(B\vec{\boldsymbol{x}})$. We can now think of the result of the product AB. We have that

$$B = \begin{bmatrix} \vec{\boldsymbol{b}}_1 & \vec{\boldsymbol{b}}_2 & \cdots & \vec{\boldsymbol{b}}_p \end{bmatrix}$$

and the product

$$B\vec{\boldsymbol{x}} = x_1\vec{\boldsymbol{b}}_1 + x_2\vec{\boldsymbol{b}}_2 + \cdots + x_p\vec{\boldsymbol{b}}_p,$$

then

$$\begin{aligned} A(B\vec{\boldsymbol{x}}) &= A\left(x_1\vec{\boldsymbol{b}}_1 + x_2\vec{\boldsymbol{b}}_2 + \cdots + x_p\vec{\boldsymbol{b}}_p\right) \\ &= x_1A\vec{\boldsymbol{b}}_1 + x_2A\vec{\boldsymbol{b}}_2 + \cdots + x_pA\vec{\boldsymbol{b}}_p \\ &= \begin{bmatrix} A\vec{\boldsymbol{b}}_1 & A\vec{\boldsymbol{b}}_2 & \cdots & A\vec{\boldsymbol{b}}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}. \end{aligned}$$

So we can say that

$$AB = \begin{bmatrix} A\vec{\boldsymbol{b}}_1 & A\vec{\boldsymbol{b}}_2 & \cdots & A\vec{\boldsymbol{b}}_p \end{bmatrix},$$

where $\vec{\boldsymbol{b}}_i$ are the columns of B.

Example 2.3.1

Perform the matrix multiplication AB where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}.$$

Using the definition of matrix multiplication, we have that

$$\begin{aligned} AB &= \begin{bmatrix} A\vec{\boldsymbol{b}}_1 & A\vec{\boldsymbol{b}}_2 & \cdots & A\vec{\boldsymbol{b}}_p \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 10 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 5+16 & 6+18 & 7+20 \\ 15+31 & 18+36 & 21+40 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{bmatrix}. \end{aligned}$$

For matrix multiplication, the number of columns of the first matrix must match the number of rows of the second matrix.

An easier way of doing matrix multiplication is to do it entry by entry. The entries of the product matrix will be

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

In other words, to find the entry in the ith row and jth column of AB, take the dot product of the ith row of A with the jth column of B.

Let A be an $m \times n$ matrix then the properties of matrix multiplication include

1. $A(BC) = (AB)C$, (associativity)
2. $A(B + C) = AB + AC$, (distributivity)
3. $(A + B)C = AC + BC$, (distributivity)
4. $r(AB) = (rA)B = A(rB)$, (associativity with scalars)
5. $I_m A = A = AI_n$, (identity)
6. $AB \neq BA$, (not commutative)
7. If $AB = AC$, then B is not necessarily equal to C.
8. If $AB = 0$, then it is possible that both $A \neq 0$ and $B \neq 0$.

Matrix multiplication isn't commutative even with square matrices.

Square $n \times n$ matrices can be raised to a power

$$A^k = A \cdot A \cdot A \cdots A.$$

At this point, we only define matrix exponentiation only for nonnegative integers. A matrix raised to the zero power is defined as the identity matrix of that size so that $A^0\vec{x} = \vec{x}$.

Tip

For matrix multiplication, the columns of the first matrix must match the rows of the second matrix. A good way to remember this to write

$$\begin{bmatrix} m \times n \end{bmatrix} \begin{bmatrix} n \times p \end{bmatrix}$$

if you're multiplying an $m\times n$ matrix by an $n\times p$ matrix. Obviously, you would replace the variables with actual numbers. Matrix multiplication is only defined if the inner pair of numbers n and n, in our case, are the same. The size of the resulting product will be an $m \times p$ matrix—the outer pair of numbers.

2.4 Transpose of a Matrix

To **transpose** a matrix is to essentially flip it about its diagonal. The transpose of A is the matrix whose columns are the rows of A. The transpose of a matrix is typically denoted with an uppercase T as in A^T.

$$\left(A^T\right)_{ij} = A_{ji}.$$

For a square matrix, the diagonal elements stay the same and the elements in the upper right move to the bottom left. The transpose of an $m \times n$ matrix will be an $n \times m$ matrix. For example,

$$\begin{bmatrix} a & b & c \\ 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} a & 1 \\ b & 2 \\ c & 3 \end{bmatrix}$$

The transpose of the product of matrices is the products of the transposed matrices in reversed order

$$\begin{aligned} (AB)^T &= B^T A^T \\ (ABC)^T &= C^T B^T A^T. \end{aligned}$$

Other properties of the transpose include

$$\begin{aligned} \left(A^T\right)^T &= A \\ (A+B)^T &= A^T + B^T \\ (rA)^T &= rA^T \end{aligned}$$

One important use of the transpose is that we can rewrite vector dot products as

$$\begin{aligned} \vec{v}\cdot\vec{u} &= \vec{v}^T\vec{u} \\ A\vec{v}\cdot\vec{u} &= (A\vec{v})^T\vec{u} = \vec{v}^T A^T\vec{u}. \end{aligned}$$

2.5 Inverse of a Matrix

An **identity matrix** is a square $n \times n$ matrix with ones along the main diagonal and zeros everywhere else. Using the Kronecker delta, we can represent this as

$$(I)_{ij} = \delta_{ij}.$$

Example identity matrices include

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

An $n \times n$ square matrix multiplied by its **inverse matrix** equals the identity matrix

$$A^{-1}A = AA^{-1} = I_n.$$

Theorem: If the **determinant** of a matrix is nonzero, then the matrix is invertible.

Keep in mind that a matrix must be square to be invertible, and that inverse matrices are unique, that is, an invertible matrix has one and only one inverse.

To find the inverse of a 2×2 matrix, use

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{\text{Det}(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $\text{Det}(A) = ad - bc$ is the determinant of A.

A matrix that is not invertible is called a **singular matrix** or a **degenerate matrix**.

Example 2.5.1

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The determinant is $\text{Det}(A) = (1)(4) - (3)(2) = -2$, so the inverse is

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Theorem: If A is an invertible $n \times n$ matrix, then for all $\vec{b}$ in $\mathbb{R}^n$, $A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

We can think of this as simply multiplying both sides of $A\vec{x} = \vec{b}$ by A^{-1}.

Example 2.5.2

Find the solution of the SLE

$$\begin{aligned} x_1 + 2x_2 &= 17 \\ 3x_1 + 4x_2 &= 39. \end{aligned}$$

We can think of this as the equation $A\vec{x} = \vec{b}$. Then then solution is $\vec{x} = A^{-1}\vec{b}$. Using the inverse we calculated in the previous example, we have that

$$\vec{x} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 17 \\ 39 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

We can also use row operations to calculate inverse matrices, but to do that, we have to look at elementary matrices. An **elementary matrix** is obtained by performing a single row operation on the identity matrix. For example,

$$E_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is the elementary matrix corresponding to the row operation "row 1 equals row 1 minus two times row three", or $R_1 = R_1 - 2R_3$ in shorthand. Similarly,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is the elementary matrix corresponding to the row operation $R_2 = 3R_2$. Elementary matrices are used to represent a row operation as a matrix multiplication. Performing a row operation on a matrix A is the same as applying the row operation to the identity matrix to get E, and then multiplying EA.

Elementary matrices are invertible. This is a consequence of the identity matrix being invertible and the fact that applying a row opertion such as $R_3 = R_3 + 4R_1$ can be reversed by applying the reverse row operation $R_3 = R_3 - 4R_1$.

We can do n row operations to a matrix A to convert it to the identity matrix

$$E_n \cdots E_2 E_1 A = I.$$

But if something times A equals the identity matrix then that something is the inverse of A. That is,

$$E_n \cdots E_2 E_1 = A^{-1}.$$

We could find A^{-1} by doing the matrix multiplication $E_n \cdots E_2 E_1$, but that would be tedious. What we do instead is apply them to the identity matrix by doing the individual row operations represented by the elementary matrices

$$E_n \cdots E_2 E_1 I = A^{-1}.$$

So to find the inverse of A, we apply elementary row operations to A until it becomes the identity matrix and we perform the same operations on the identity matrix to get the inverse of A. To do this, we create an augmented matrix of A and I. Now any row operation that is applied to A is also being applied to the identity matrix at the same time. When the A side of the augmented matrix has been reduced to row echelon form (i.e. turned into the identity matrix), the other side will be A^{-1}.

Example 2.5.3

Find the inverse of

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 8 & 1 \\ 5 & 4 & 3 \end{bmatrix}.$$

To find the inverse, we form an augmented matrix of A and I_3.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 4 & 1 & 0 & 0 \\ 6 & 8 & 1 & 0 & 1 & 0 \\ 5 & 4 & 3 & 0 & 0 & 1 \end{array}\right].$$

Then we perform elementary row operations to convert the left half of the augmented matrix into the identity matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & \frac{13}{30} & \frac{11}{30} & -\frac{7}{10} \\ 0 & 0 & 1 & \frac{8}{15} & \frac{1}{15} & -\frac{2}{5} \end{array}\right].$$

Now we know that the inverse is

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & 1 \\ \frac{13}{30} & \frac{11}{30} & -\frac{7}{10} \\ \frac{8}{15} & \frac{1}{15} & -\frac{2}{5} \end{bmatrix}.$$

If A is an invertible $n \times n$ matrix, then all of the following statements are equivalent:

1. A is invertible
2. $AA^{-1} = I_n$, and $A^{-1}A = I_n$
3. The reduced echelon form of A is I_n
4. A and I are row equivalent
5. A has n pivot positions
6. A has no free variables
7. $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ has only the trivial solution
8. The columns of A are linearly independent
9. $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is one-to-one
10. For every $\vec{\boldsymbol{b}} \in \mathbb{R}^N$, $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution
11. The columns of A span $\mathbb{R}^n$
12. $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is onto
13. A^T is invertible

The proof that A^T is invertible if A is invertible is simple and uses the fact that

$(AB)^T = B^T A^T$

$$\begin{aligned} AA^{-1} &= I \quad \text{by definition} \\ (AA^{-1})^T &= I^T \quad \text{transpose both sides} \\ (AA^{-1})^T &= I \quad \text{transpose of } I \text{ is } I \\ (A^{-1})^T A^T &= I^T \quad \text{by } (AB)^T = B^T A^T. \end{aligned}$$

Tip

One of the easiest ways to tell if a matrix is invertible is to put it into echelon form and see if there are any free variables.

This shows that $(A^{-1})^T = (A^T)^{-1}$.

Theorem: A transformation $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is invertible if A is invertible. Then $S = T^{-1}$ is given by $S = A^{-1}\vec{\boldsymbol{x}}$.

A transformation is invertible if there exists a transformation that reverses its effect on a vector. If $S(T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}$ and $T(S(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}$, then T and S are inverse transformations. By the theorem above, the inverse of T is just the transformation that uses the inverse of the standard matrix that T uses.

2.6 Partitioned Matrices

We have previously considered a matrix as a queue of column vectors. A matrix that is divided into submatrices like that is a **partitioned matrix** or **block matrix**.

Matrices can be partitioned in any way provided that the lines are horizontal or vertical (no diagonal partitions) and any line crosses the entire matrix.

If two partitioned matrices A and B are of the same size and have the same partitioning, then the addition $A+B$ and scalar multiplication cA can be performed block by block.

Things are a little more complicated with matrix multiplication. Consider the partioned matrices

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 3 & -2 & -1 \\ 2 & 1 & 0 & 1 & 2 \\ \hline -2 & 4 & 1 & -2 & 3 \end{array}\right], \quad B = \left[\begin{array}{cc} -2 & -1 \\ 1 & 2 \\ 4 & 1 \\ \hline 2 & 3 \\ 1 & 0 \end{array}\right].$$

We can think of these matrices as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 & -1 \\ 1 & 2 \\ 4 & 1 \end{bmatrix},$$

and so on. Then to multiply AB, we can use the partioned versions and do the multiplication block by block

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}.$$

Finally, we calculate the products of the submatrices such as $A_{11}B_1$ and plug those values in above. Obviously, in order for matrix multiplication to work the column partition of A must match the row partition of B.

One convenient way of partitioning that always works is to partition A into column vectors and B into row vectors, then AB is a simple matter of multiplying a column and a row vector a bunch of times. Doing this, then in general, if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then

$$\begin{aligned} AB &= \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \\ &= \sum_{k=1}^{n} \mathrm{Col}_k(A)\ \mathrm{Row}_k(B). \end{aligned}$$

In other words, just multiply the kth column of A times the kth row of B and add them all up.

Partitioned matrices don't reduce the work in matrix multiplication, but they can help with organization or when performing extremely large matrix multiplications on a computer.

A partitioned matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is called a **block upper triangular** matrix. Note that A_{ij} is a submatrix. The diagonal submatrices must be square. If A_{11} is $p \times p$ and A_{22} is $q \times q$, then A_{12} must be $p \times q$, and the overall matrix will be $(p+q) \times (p+q)$.

We want to find the inverse of this block upper triangular matrix. Recall that $AB = I$, so we can write

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_{11} & 0 \\ 0 & I_{22} \end{bmatrix}.$$

Doing the matrix multiplication shows us that

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I_{11} \\ A_{11}B_{12} + A_{12}B_{22} &= 0 \\ 0 + A_{22}B_{21} &= 0 \\ 0 + A_{22}B_{22} &= I_{22} \end{aligned}$$

Notice with the third equation that since A_{22} is not zero then $B_{21} = 0$. From the fourth equation, notice that $B_{22} = A_{22}^{-1}$. For the second equation, if we move the second term to the right side then multiply both sides by A_{11}^{-1} we get that $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$. Since we know that $B_{21} = 0$, we can plug this into the first equation to find that $B_{11} = A_{11}^{-1}$. So B, the inverse of A is given by

$$B = A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

If we're given a block upper triangular matrix, then provided that A_{11} and A_{22} are invertible, we can just use the above as a formula for finding A^{-1}.

2.7 Leontief Input-Output Model

In an economy, there are producers and consumers. However, the producers are also consumers. For example, the manufacturing producer might need to consume some of the chemical supplies producer's product in order to manufacture stuff. This stuff that is needed by the producers is called the **intermediate demand**. There's also a final demand–that from the pure consumers that are not also producers. We want to be able to figure out the total manufacturing output, keeping in mind that the total demand is larger than the final demand due to the fact that the producers have to consume in order to produce.

$$\begin{matrix}\text{total} \\ \text{produced}\end{matrix} = \begin{matrix}\text{intermediate} \\ \text{demand}\end{matrix} + \begin{matrix}\text{final} \\ \text{demand}\end{matrix}$$

We can write this as

$$\vec{x} = C\vec{x} + \vec{d},$$

where $C\vec{x}$ is the intermediate demand, and $\vec{d}$ is the final demand. What we want to solve for is $\vec{x}$, but notice that it appears on both sides of the equation. The quantity $C\vec{x}$ is some fraction of the total product $\vec{x}$ and this implies that C is less than one, that is, its entries are less than one. Specifically, C is the consumption matrix that gives the inputs consumed by the producers per unit of output.

Consider for example an economy consisting of the three sectors: food, transportation, and manufacturing. Obviously, some food, some transportation, and some manufacturing will be consumed for every unit of manufacturing that is produced. All the sectors depend on each other. We can construct the consumption matrix by considering the inputs consumed by the producers per unit of output.

	Inputs Consumed per Unit of Output		
Purchased From:	Food	Transportation	Manufacturing
Food	0.1	0.2	0.2
Transportation	0.4	0.3	0.5
Manufacturing	0.2	0.5	0.4

This table shows us, for example, that the food sector consumes 0.1 units of food, 0.4 units of transportation, and 0.2 units of manufacturing for every unit of food that is produced. From this table, we have that our consumption matrix is

$$C = \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.5 \\ 0.2 & 0.5 & 0.4 \end{bmatrix}.$$

If the final demand is 50 units of food, 40 units of transportation, and 20 units of manufacturing, what is the production level that will satisfy the demand?

We can rewrite our equation by making use of the fact that $\vec{\boldsymbol{x}} = I\vec{\boldsymbol{x}}$ to write

$$(I - C)\vec{\boldsymbol{x}} = \vec{\boldsymbol{d}}.$$

Subtracting C from the 3×3 identity matrix gives us

$$I - C = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ -0.4 & 0.7 & -0.5 \\ -0.2 & -0.5 & 0.6 \end{bmatrix}.$$

Our augmented matrix is then

$$\left[\begin{array}{ccc|c} 0.9 & -0.2 & -0.2 & 50 \\ -0.4 & 0.7 & -0.5 & 40 \\ -0.2 & -0.5 & 0.6 & 20 \end{array}\right]$$

Putting it into reduced echelon form gives us

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1300 \\ 0 & 1 & 0 & 2800 \\ 0 & 0 & 1 & 2800 \end{array}\right].$$

So the total production needs to be 1300 units of food, and 2800 units each of transportation and manufacturing.

Another way we could solve the problem is by multiplying both sides of the earlier equation by $(I - C)^{-1}$ to get

$$\vec{\boldsymbol{x}} = (I - C)^{-1}\vec{\boldsymbol{d}}.$$

This requires us to find the inverse of the matrix $(I - C)$, but it's useful in that we can use the same matrix to solve the problem for various $\vec{\boldsymbol{d}}$. For example, if the final demand $\vec{\boldsymbol{d}}$ is changing from year to year, we could still use the same inverse and in the long run, it would be easier.

Suppose the manufacturers decide to produce exactly enough to meet the final demand $\vec{\boldsymbol{x}} = \vec{\boldsymbol{d}}$. As soon as they begin, they realize that they need product from each other to fulfill the final demand. This additional demand is $C\vec{\boldsymbol{d}}$. But in order to produce enough for this additional demand, they need still more product $C(C\vec{\boldsymbol{d}}) = C^2\vec{\boldsymbol{d}}$ and so on. So the total that needs to be produced is

$$\begin{aligned} \vec{\boldsymbol{x}} &= \vec{\boldsymbol{d}} + C\vec{\boldsymbol{d}} + C^2\vec{\boldsymbol{d}} + C^3\vec{\boldsymbol{d}} + \cdots \\ &= \left(I + C + C^2 + C^3 + \cdots\right)\vec{\boldsymbol{d}}. \end{aligned}$$

But earlier we found that $\vec{x} = (I - C)^{-1}\vec{d}$, so it must be the case that

$$(I - C)^{-1} = \left(I + C + C^2 + C^3 + \cdots\right).$$

So to find $(I - C)^{-1}$ we can

1. Create the augmented matrix $[(I - C) \,|\, I]$ and row reduce to get the inverse of $(I - C)$, or
2. Approximate it as $I + C + C^2 + \cdots + C^n$ using a large n. Since all the entries of C are less than 1, higher powers of C will be much smaller than C.

2.8 Subspace

A **subspace** is a special kind of subset. W is a subspace of $\mathbb{R}^n$ if and only if all of the following are satisfied

1. $\vec{0}$ is in W
2. If $\vec{u}$ and $\vec{v}$ are in W then $\vec{u} + \vec{v}$ is in W
3. If $\vec{u}$ is in W, then $c\vec{u}$ is in W

Items two and three imply closure under linear combination. In other words, for W to be a subspace, then any linear combination of vectors in W must also be in W.

To verify a subspace, check that it contains the zero vector and that all linear combinations of vectors in the subspace are also in the subspace.

Examples of subspaces include

- $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$
- A plane through the origin in $\mathbb{R}^3$ is a subspace
- A line through the origin in $\mathbb{R}^3$ is a subspace
- A line through the origin in $\mathbb{R}^2$ is a subspace

Example 2.8.1

If $\vec{v}_1$ and $\vec{v}_2$ are in $\mathbb{R}^n$, is $\text{span}(\vec{v}_1, \vec{v}_2)$ a subspace of $\mathbb{R}^n$?

We know that the zero vector is contained in $\text{span}(\vec{v}_1, \vec{v}_2)$ because $\vec{0} = (0)\vec{v}_1 + (0)\vec{v}_2$. To see if all linear combinations of $\vec{v}_1$ and $\vec{v}_2$ are also in $\text{span}(\vec{v}_1, \vec{v}_2)$, we'll start with any two vectors from $\text{span}(\vec{v}_1, \vec{v}_2)$, say $\vec{u}$ and $\vec{w}$, and it must be that $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{w} = c_3\vec{v}_1 + c_4\vec{v}_2$ where c_i are arbitrary constants. Therefore,

$$\begin{aligned} c\vec{u} + d\vec{w} &= c(c_1\vec{v}_1 + c_2\vec{v}_2) + d(c_3\vec{v}_1 + c_4\vec{v}_2) \\ &= (cc_1 + dc_3)\vec{v}_1 + (cc_2 + dc_4)\vec{v}_2. \end{aligned}$$

Since $(cc_1 + dc_3)$ and $(cc_2 + dc_4)$ are just numbers, this shows that $\vec{u}$ and $\vec{w}$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$. Therefore, $\text{span}(\vec{v}_1, \vec{v}_2)$ is a subspace of $\mathbb{R}^n$.

It turns out that $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{p})$ is a subspace of $\mathbb{R}^n$, and so we can just call $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{p})$ the subspace formed by the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{p}$. The proof of this is just an extension of the proof given in the example above.

The **column space** of a matrix A, denoted Col A, is the set of all linear combinations of the columns of A. In other words, Col $A = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, where $\vec{a}_i$ is the ith column of A. So Col A is a subspace.

For the transformation $T(\vec{x}) = A\vec{x}$, the range of T is the same as Col A since multiplying A by all possible $\vec{x}$ gives the linear combination of the columns of A.

To determine if a given vector $\vec{b}$ is in Col A, just create an augmented matrix $\left[A \,|\, \vec{b}\right]$ and reduce it to echelon form. Remember, you're just checking to see if $\vec{b}$ is in the span of the columns of A.

The **null space** of a matrix A, denoted Nul A, is the set of all vectors $\vec{\boldsymbol{x}}$ such that $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$. We know that $\vec{\mathbf{0}}$ is in Nul A, and for $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{w}}$ in Nul A, we have that

$$\begin{aligned} A(c\vec{\boldsymbol{u}} + d\vec{\boldsymbol{v}}) &= \vec{\mathbf{0}} \\ Ac\vec{\boldsymbol{u}} + Ad\vec{\boldsymbol{v}} &= \vec{\mathbf{0}} \\ c(A\vec{\boldsymbol{u}}) + d(A\vec{\boldsymbol{v}}) &= \vec{\mathbf{0}} \\ c(\vec{\mathbf{0}}) + d(\vec{\mathbf{0}}) &= \vec{\mathbf{0}}, \end{aligned}$$

and so $c\vec{\boldsymbol{u}} + d\vec{\boldsymbol{v}}$ is also in Nul A. Therefore, Nul A is a subspace.

The **basis of a subspace** W of $\mathbb{R}^n$ is a linearly independent set of vectors in W that spans W. This can be thought of as the largest set of independent vectors in W or as the smallest set of vectors in W that spans W. For example, the standard basis vectors

$$\vec{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{\boldsymbol{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is a basis of $\mathbb{R}^3$ since the three vectors are linearly independent and they span $\mathbb{R}^3$. Incidentally, $\mathbb{R}^3$ is itself a subspace of $\mathbb{R}^3$.

If a square $n \times n$ matrix is invertible, we know the columns are linearly independent. If the matrix is invertible, then you can transform it into the identity matrix by applying row operations, therefore, it can't have any free variables. Such a matrix also necessarily spans $\mathbb{R}^n$. This means that the columns of any invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

Example 2.8.2

Find a basis for Nul A given

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 5 \\ 2 & 4 & 6 & 1 \end{bmatrix}.$$

Remember that Nul A is the set of $\vec{\boldsymbol{x}}$ such that $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$. Forming the augmented matrix and reducing it to reduced echelon form gives us

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Our solution for $A\vec{x} = \vec{0}$ is

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since any linear combinations of the above two vectors satisfy $A\vec{x} = \vec{0}$, the span of the two vectors is the same as Nul A. Since the two vectors are span Nul A and they are linearly independent, the same vectors form a basis for Nul A. This is not the only possible basis for Nul A. For example, we could scale the two by any constants, and they would still form a basis for Nul A.

From the example above, we know that to find a basis for Nul A,

1. Find the set of all $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$. To do this, simply solve the augmented matrix $\left[A \,|\, \vec{0}\right]$.
2. From those vectors, select the smallest linearly independent set that spans the entire set.

Example 2.8.3

Find a basis for Col A given

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 5 \\ 2 & 4 & 6 & 1 \end{bmatrix}.$$

The RREF of A is

$$B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that columns two and three are linear combinations of the first column (in this case just scalar multiples), so the column span of B is span$\{\vec{b}_1, \vec{b}_4\}$. These two vectors are also clearly linearly independent, so span$\{\vec{b}_1, \vec{b}_4\}$ is a basis for Col B. Notice that $\vec{b}_1$ and $\vec{b}_2$ are the pivot columns of B. In general, for a matrix in RREF, the pivot columns form a basis for the column space of the matrix.

The columns of A have the same dependence relations as the columns of B if A and B are row equivalent. So the pivot columns of A, $\{\vec{a}_1, \vec{a}_4\}$, form a basis for Col A. So a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Notice that $\{\vec{a}_1, \vec{a}_4\}$ is a different subspace than $\{\vec{b}_1, \vec{b}_4\}$ despite the two matrices being row equivalent. Notice that the third component in $\{\vec{b}_1, \vec{b}_4\}$ is zero for both vectors, so span$\{\vec{b}_1, \vec{b}_4\}$ is the plane formed by x_1 and x_2.

From the example above, we know that to find a basis for Col A,

1. Put A into reduced echelon form
2. Identity the pivot columns
3. Write down the pivot columns of the original matrix (before it was transformed into RREF) as a basis for Col A

Every vector has a unique representation under a chosen basis. For example, the vector

$$\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

under the standard basis for $\mathbb{R}^3$ can be written as a linear combination of the standard basis vectors only as

$$3\vec{e}_1 + 5\vec{e}_2 + \vec{e}_3.$$

The fact that a vector has a unique representation under a basis is why we choose a basis instead of just any spanning set.

If we have an ordered set of basis vectors $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$, then a vector $\vec{x}$ under that basis, is represented as

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n.$$

We say that

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **B-coordinate vector of $\vec{x}$**. To find the B-coordinate vector of $\vec{x}$ given some $\vec{x}$ and some ordered basis B, you have to determine how $\vec{x}$ is represented in the basis B and then use the coefficients obtained there to construct the B-coordinate vector of $\vec{x}$.

Example 2.8.4

Find the B-coordinate vector of $\vec{x}$ given the basis $B = \{\vec{b}_1, \vec{b}_2\}$.

$$\vec{x} = \begin{bmatrix} 8 \\ 19 \\ 9 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Since $\{\vec{b}_1, \vec{b}_2\}$ are basis vectors, we know that they span a plane. The vector $\vec{x}$ must lie in the same plane in order to be written as a linear combination of the basis vectors. We are asked to find the coefficients c_i such that $\vec{x}$ can be written as the linear combination

$$\begin{bmatrix} 8 \\ 19 \\ 9 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}.$$

Creating the augmented matrix and transforming it into RREF, we have that

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 2 & 5 & 19 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

So our coefficients are $c_1 = 2$ and $c_2 = 3$, and the B-coordinate vector of $\vec{x}$ is

$$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Notice that there are the same number of elements in the B-coordinate vector of $\vec{x}$ as there are basis vectors in B. This is true in general.

The **dimension** of a nonzero subspace W, called Dim W, is the number of vectors in any basis of W. For example, Dim $\mathbb{R}^n = n$ and $\{\vec{0}\}$ is a subspace with dimension 0.

The **rank** of a matrix is the dimension of its column space

$$\text{Rank } A = \text{Dim Col } A.$$

Recall that the pivot columns of a matrix form a basis for the column space of the matrix. Therefore, the rank of a matrix is the same as its number of pivot columns since the basis for its column space will have that number of vectors.

Theorem: If a matrix A has n columns then Rank $A +$ Dim Nul $A = n$.This makes sense because the nonpivot columns of A correspond to free variables in $A\vec{x} = \vec{0}$, and for every one of these free variables, there will be a vector in Nul A. So the number of vectors in Col A plus the number of vectors in Nul A equals the number of columns of A.

Theorem: If Dim $W = p$, where W is a subspace of $\mathbb{R}^n$, then

- Any collection of p linearly independent vectors in W is a basis for W, and
- Any collection of p vectors that span W is a basis for W.

So if either one of these is true, we have a basis. We only need to check one of them.

If A is an $n \times n$ matrix, then the following statements are equivalent

1. A is invertible
2. The columns of A are independent
3. The columns of A are a basis for $\mathbb{R}^n$
4. Col $A = \mathbb{R}^n$, that is, the columns span $\mathbb{R}^n$
5. Dim Col $A =$ Rank $A = n$
6. Dim Nul $A = 0$
7. Nul $A = \{\vec{\mathbf{0}}\}$

2.9 Summary: Matrix Algebra

Properties of matrix multiplication include

1. $A(BC) = (AB)C$, (associativity)
2. $A(B + C) = AB + AC$, (distributivity)
3. $(A + B)C = AC + BC$, (distributivity)
4. $r(AB) = (rA)B = A(rB)$, (associativity with scalars)
5. $I_m A = A = AI_n$, (identity)
6. $AB \neq BA$, (not commutative).

Properties of the transpose include

$$\begin{aligned} (AB)^T &= B^T A^T \\ \left(A^T\right)^T &= A \\ (A+B)^T &= A^T + B^T \\ (rA)^T &= rA^T. \end{aligned}$$

To determine if a matrix is invertible

1. Put it into echelon form. If there are free variables, the matrix is not invertible.
2. If the determinant is zero, the matrix is not invertible.

For a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To invert a matrix A, you can use matrix transformations on the augmented matrix containing the identity matrix on the right side $\left[A \,|\, I \right] \to \left[I \,|\, A^{-1} \right]$.

If A is an $n \times n$ matrix, then by the **invertible matrix theorem**, the following statements are equivalent:

1. A is invertible
2. $AA^{-1} = I_n$, and $A^{-1}A = I_n$
3. The reduced echelon form of A is I_n
4. A has n pivot positions
5. A has no free variables
6. $A\vec{x} = \vec{0}$ has only the trivial solution
7. The columns of A are linearly independent
8. $T(\vec{x}) = A\vec{x}$ is one-to-one
9. For every $\vec{b} \in \mathbb{R}^N$, $A\vec{x} = \vec{b}$ has a unique solution
10. The columns of A span $\mathbb{R}^n$
11. $T(\vec{x}) = A\vec{x}$ is onto
12. A^T is invertible.

For the **Leontiff input-output model**, we will typically be given the input-output matrix C and the final demand $\vec{d}$, and be asked to determine the necessary production $\vec{x}$. The form of the equation is

$$\vec{x} = C\vec{x} + \vec{d}.$$

We can transform the equation to

$$(I - C)\vec{x} = \vec{d},$$

and solve using the augmented matrix or to

$$\vec{x} = (I - C)^{-1}\vec{d},$$

calculate the inverse and then perform the multiplication. The first method is typically easier.

A **subspace** is a special kind of subset. W is a subspace of $\mathbb{R}^n$ if and only if all of the following are satisfied

1. $\vec{0}$ is in W
2. If $\vec{u}$ and $\vec{v}$ are in W then $\vec{u} + \vec{v}$ is in W
3. If $\vec{u}$ is in W, then $c\vec{u}$ is in W

To verify a subspace, check that it contains the zero vector and that all linear combinations of vectors in the subspace are also in the subspace.

The **column space** of a matrix A, denoted Col A, is the set of all linear combinations of the columns of A. In other words, Col $A = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, where $\vec{a}_i$ is the ith column of A. To determine if a given vector $\vec{b}$ is in Col A, just create an augmented matrix $\left[A \,|\, \vec{b} \right]$ and reduce it to echelon form. Remember, you're just checking to see if $\vec{b}$ is in the span of the columns of A.

The **null space** of a matrix A, denoted Nul A, is the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

The **basis of a subspace** W of $\mathbb{R}^n$ is a linearly independent set of vectors in W that spans W. This can be thought of as the largest set of independent vectors in W.

To find a basis for Nul A,

1. Find the set of all $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$. To do that, simply solve the augmented matrix $\left[A \,|\, \vec{0} \right]$.
2. From those vectors, select the smallest linearly independent set that spans the entire set.

To find a basis for Col A,

1. Put A into reduced echelon form
2. Identity the pivot columns
3. Write down the pivot columns of the original matrix (before it was transformed into RREF) as a basis for Col A

If we have an ordered set of basis vectors $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$, then a vector $\vec{x}$ under that basis, is represented as $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$. We say that

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the B-**coordinate vector of** $\vec{\boldsymbol{x}}$. To find the B-coordinate vector of $\vec{\boldsymbol{x}}$ given some $\vec{\boldsymbol{x}}$ and some ordered basis B, you have to determine how $\vec{\boldsymbol{x}}$ is represented in the basis B and then use the coefficients obtained there to construct the B-coordinate vector of $\vec{\boldsymbol{x}}$.

The **dimension** of a nonzero subspace W, called Dim W, is the number of vectors in any basis of W. For example, Dim $\mathbb{R}^n = n$ and $\{\vec{\mathbf{0}}\}$ is a subspace with dimension 0.

The **rank** of a matrix is the dimension of its column space

$$\text{Rank } A = \text{Dim Col } A.$$

Recall that the pivot columns of a matrix form a basis for the column space of the matrix. Therefore, the rank of a matrix is the same as its number of pivot columns since the basis for its column space will have that number of vectors.

If a matrix A has n columns then Rank A + Dim Nul $A = n$.This makes sense because the nonpivot columns of A correspond to free variables in $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$, and for every one of these free variables, there will be a vector in Nul A. So the number of vectors in Col A plus the number of vectors in Nul A equals the number of columns of A.

For an $n \times n$ matrix A, by the **invertible matrix theorem**, the the following statements are equivalent

1. A is invertible
2. The columns of A are independent
3. The columns of A are a basis for $\mathbb{R}^n$
4. Col $A = \mathbb{R}^n$, that is, the columns span $\mathbb{R}^n$
5. Dim Col A = Rank $A = n$
6. Dim Nul $A = 0$
7. Nul $A = \{\vec{\mathbf{0}}\}$

Chapter 3

Determinants

If A is an $n \times n$ matrix, then $\det A$ is the determinant (a scalar) of A.

Theorem: If $\det A = 0$, then A is not invertible. If $\det A \neq 0$, then A is invertible. This is an important theorem that needs to be memorized.

If A is the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then putting A into echelon form by performing the row operation $R_2 = aR_2 - cR_1$ gives us

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}.$$

Recall that if A is an invertible matrix, for for each $\vec{\boldsymbol{b}}$ in $\mathbb{R}^2$, the equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution. In this case, we know that if $ad - bc = 0$, then A is not invertible because if we made the augmented matrix $\left[A \,|\, \vec{\boldsymbol{b}}\right]$, then for any $\vec{\boldsymbol{b}} \neq \vec{\boldsymbol{0}}$ there would not be a solution. We call the quantity $ad - bc$ the **determinant** of A denoted

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The **minor** M_{ij} of A is the determinant of the submatrix obtained by deleting row i and column j from matrix A. For example, in the 3×3 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

the minor M_{12} is

$$M_{12} = \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} = A_{21}A_{33} - A_{23}A_{31}.$$

A **cofactor** C_{ij} is a signed minor obtained by multiplying the minor M_{ij} by negative one if the sum of the indices $i + j$ is odd.

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is a minor. For example,

$$C_{21} = (-1)^{2+1} M_{ij} = -M_{ij}.$$

Laplace expansion allows us to compute the determinants of large matrices by expressing the large determinant in terms of smaller ones. With this method, the determinant of the large matrix can be "expanded" along any row or column.

Using Laplace expansion, the determinant of an $n \times n$ matrix A expanded along the ith row can be computed as

$$\det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} = \sum_{j=1}^{n} A_{ij}C_{ij}.$$

Similarly, expanding along the jth column can be done by computing

$$\det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj} = \sum_{i=1}^{n} A_{ij}C_{ij}.$$

In other words, to expand along a row, you just sum over the row multiplying the row elements by the corresponding cofactors. Similarly, to expand along a column, you just sum down the column, multiplying the column elememts by the corresponding cofactors.

To calculate the determinant of a large matrix, you have to calculate many nested determinants. If A is an $n \times n$ matrix, then to calculate $\det A$ you actually have to calculate $\frac{1}{2}n!$ determinants.

Example 3.0.1

Calculate the determinant of A if

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

We can calculate the determinant by expanding along any row or column, but for convenience, we choose the row or column with the most zeros since this will result in the most simple expansion. In our case, they all have three zeros, so we'll expand along the first row. The only nonzero element is A_{12}, which has the cofactor $(1)(-1)^{1+2}M_{12} = -M_{12}$, so our determinant is

$$\det(A) = (1)(-1)^{1+2}\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}.$$

If we expand this along the first row, the only non-zero term is the first one which gives us $(-1)(-1)^{1+1}M_{11} = -M_{11}$, so our determinant is

$$\det(A) = (1)(-1)^{1+2}(-1)(-1)^{1+1}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1.$$

Notice that for a 2×2 matrix A, we can also compute the determinant as

$$\det(A) = \sum_{i,j=1}^{2} \varepsilon_{ij} A_{i1} A_{j2},$$

and the determinant of a 3×3 matrix A as

$$\det(A) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} A_{i1} A_{j2} A_{k3},$$

where ε_{ijk} is the totally antisymmetric Levi-Cevita symbol and ε_{ij} is similarly antisymmetric. So the determinant is the antisymmetric combination of products A_{jk} of a matrix A.

Properties of determinants:

1. Interchanging or swapping any two rows or columns causes the determinant to change sign.
2. The determinant is unchanged if a multiple of a row (or column) is added to a row (or column).
3. The determinant is zero if the columns (or rows) of a matrix are linearly dependent. So if any pair of rows or pair of columns is the same or if any row or column is all zeros then the determinant is zero. This is obvious from the formula for Laplace expansion. Since you can expand along any row or column, you would pick the one with all zeros, then every term in the expansion would have a factor of zero. This property makes it easy to determine if n, n-dimensional vectors are linearly independent or not. Just put them in a matrix. If the determinant of the matrix is nonzero then the vectors are linearly independent.
4. The determinant of the inverse of a matrix is the inverse of the determinant of the matrix
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$
From this property, it follows that
$$\det(A) \cdot \det(A^{-1}) = 1.$$
5. The determinant of a matrix is equal to the determinant of the transpose of the matrix
$$\det(A^T) = \det(A).$$
One result of this property is that we can do column operations to compute $\det A$ with the same rules that we use when doing row operations. We can also mix and match row and column operations.
6. Multiplying a row or a column by a constant changes the determinant by the same factor. This means that if the entire matrix is multiplied by a factor than the determinant is multiplied by the factor raised to the power of the number of rows (or columns). For an $n \times n$ matrix A
$$\text{If } \det(A) = d, \text{ then } \det(\alpha A) = \alpha^n d.$$
This property allows you to simplify determinants by factoring out a constant and raising that constant to the power of the number of rows in the determinant.
7. The determinant of a product of matrices is the product of the determinants of the matrices
$$\det(AB) = \det(A) \cdot \det(B).$$
8. The determinant of a matrix in triangular form (i.e. only zeros below the diagonal) is the product of the diagonal elements. From this it follows that the determinant of all identity matrices is one
$$\det(I) = 1.$$

9. If you change a row in a matrix by adding some constants, you can calculate the determinant as the sum of the determinant of the original matrix and the determinant of the same matrix with that row replaced by the constants added to the original matrix. That is, we can write

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g+j & h+k & i+l \end{vmatrix},$$

as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ j & k & l \end{vmatrix}$$

The Laplacian expansion method proves tedious for large matrices, so a second way to calculate determinants is to make use of their properties. Here's another way to calculate a determinant:

1. Using elementary row operations, transform a matrix A into a matrix B which is in echelon form, keeping track of all row operations performed.
2. $\det B$ is now the product of the diagonal entries.
3. Look at each row operation that was performed to transform A into B. The determinant of A is related to the determinant of B via the effect that those row operations have on changing the determinant.
   - $R_i = kR_i \implies k \det A = \det B$, that is, if you multiply a row in A by a scalar to get B, then you have to divide $\det B$ by that scalar in order to get $\det A$.
   - $R_i = R_i + kR_i \implies \det A = \det B$, that is the row operations that involve adding a multiple of a row to another row do not change the determinant.
   - Swapping R_i and $R_j \implies -\det A = \det B$, that is, for every row swap, change the sign of the determinant of B to get the determinant of A.

If you performed multiple row operations to A to get B, you just perform the relevant adjustment to $\det B$ for each row operation performed to get $\det A$.

3.1 Cramer's Rule

Consider the matrix equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$. Notice that $A\vec{\boldsymbol{e}}_i = \vec{\boldsymbol{a}}_i$. That is, A times the ith column of the identity matrix, gives the ith column of A. Replacing the ith column of A by $\vec{\boldsymbol{b}}$ denoted $A_i(\vec{\boldsymbol{b}})$ can be written as

$$\begin{aligned} A_i(\vec{\boldsymbol{b}}) &= \begin{bmatrix} A\vec{\boldsymbol{e}}_1 \cdots A\vec{\boldsymbol{x}} \cdots A\vec{\boldsymbol{e}}_n \end{bmatrix} \\ &= A\begin{bmatrix} \vec{\boldsymbol{e}}_1 \cdots \vec{\boldsymbol{x}} \cdots \vec{\boldsymbol{e}}_n \end{bmatrix}. \end{aligned}$$

Taking the determinant of both sides, we have that

$$\det A_i(\vec{\boldsymbol{b}}) = \det A \cdot \det \begin{bmatrix} \vec{\boldsymbol{e}}_1 \cdots \vec{\boldsymbol{x}} \cdots \vec{\boldsymbol{e}}_n \end{bmatrix}.$$

Notice that if A is a 3×3 matrix and if we replace the second column, then

$$\det A \cdot \det \begin{bmatrix} \vec{\boldsymbol{e}}_1 \cdots \vec{\boldsymbol{x}} \cdots \vec{\boldsymbol{e}}_n \end{bmatrix} = \det \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}.$$

Notice that the matrix on the right is formed by applying row operations to the identity matrix. Specifically, the row operations $R_1 = R_1 + x_1R_2$, and $R_2 = x_2R_2$, and $R_3 = R_3 + x_3R_3$ are performed. The determinant of I is 1, and the only one of these three row operations that changes the determinant is $R_2 = x_2R_2$. So, the determinant on the right in the equation above is just x_2. This will always be the case—the determinant will be x_i, where i is the column that was replaced. In other words,

$$\det A_i(\vec{\boldsymbol{b}}) = \det A \cdot x_i,$$

and rearranging, gives us Cramer's rule.

Suppose that A is an $n \times n$ matrix and $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, then **Cramer's rule** states that

$$x_i = \frac{\det A_i(\vec{\boldsymbol{b}})}{\det A} = \frac{\left|\vec{\boldsymbol{a}}_1 \cdots \vec{\boldsymbol{b}} \cdots \vec{\boldsymbol{a}}_n\right|}{|A|},$$

where x_i is the ith component of $\vec{\boldsymbol{x}}$ and $A_i(\vec{\boldsymbol{b}})$ is the matrix A with the ith column replaced by $\vec{\boldsymbol{b}}$. Notice that Cramer's rule won't work if $\det A = 0$ (i.e. if A is not invertable).

Example 3.1.1

Find $\vec{\boldsymbol{x}}$ in $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ if $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \end{bmatrix}.$$

Calculating the determinant of A, we have that $\det A = (2)(4) - (1)(3) = 5$. And

$$A_1(\vec{\boldsymbol{b}}) = \begin{bmatrix} 11 & 1 \\ 29 & 4 \end{bmatrix}, \quad A_2(\vec{\boldsymbol{b}}) \begin{bmatrix} 2 & 11 \\ 3 & 29 \end{bmatrix},$$

with determinants $\det A_1(\vec{\boldsymbol{b}}) = 15$, and $\det A_2(\vec{\boldsymbol{b}}) = 25$, so

$$x_1 = \frac{15}{5} = 3, \quad x_2 = \frac{25}{5} = 5,$$

so

$$\vec{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Using Cramer's rule to find $\vec{\boldsymbol{x}}$ requires that you calculate $n + 1$ determinants of size $n \times n$ just to calculate a single component of $\vec{\boldsymbol{x}}$, so it's not an efficient method. The power of Cramer's rule shows when you have matrices containing functions (e.g. Jacobians). With functions, we can't easily use row operations to eliminate stuff, but we can use Cramer's rule.

Example 3.1.2

Choose R so that $A\vec{x} = \vec{b}$ has a unique solution if

$$A = \begin{bmatrix} R & -3 \\ -4 & 2R \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -14 \\ 12 \end{bmatrix},$$

and find a formula for the solution.

In order for $A\vec{x} = \vec{b}$ to have a unique solution, $\det A \neq 0$. Calculating the determinant, we have that $\det A = 2R^2 - 12 = 2(R^2 - 6) = 2(R - \sqrt{6})(R + \sqrt{6})$. So in order for it to have a unique solution, $R \neq \pm\sqrt{6}$.

to find the actual form of the solution, we use Cramer's rule. Because

$$A_1(\vec{b}) = \begin{bmatrix} -14 & -3 \\ 12 & 2R \end{bmatrix}, \quad A_2(\vec{b}) \begin{bmatrix} R & -14 \\ -4 & 12 \end{bmatrix},$$

we have that $\det A_1(\vec{b}) = -28R + 36 =$ and $\det A_2(\vec{b}) = 12R - 56$, so

$$x_1 = \frac{18 - 14R}{R^2 - 6}, \quad x_2 = \frac{6R - 28}{R^2 - 6}.$$

So the formula for our solution is

$$\vec{x} = \begin{bmatrix} \frac{18-14R}{R^2-6} \\ \frac{6R-28}{R^2-6} \end{bmatrix}.$$

We can also calculate the inverse of a matrix using Cramer's rule. This is especially useful if we have variables in the matrix.

If A is an $n \times n$ matrix, and $A\vec{x} = \vec{e}_j$, where $\vec{e}_j$ is the jth column of the identity matrix, then we can multiply both sides by the inverse of A to get $\vec{x} = A^{-1}\vec{e}_j$. Recall that multiplying a matrix by the jth column of the identity matrix gives the jth column of the matrix. In our case, $\vec{x}$ is the jth column of A^{-1}. We can solve for $\vec{x}$ using Cramer's rule.

$$(A^{-1})_{ij} = x_i = \frac{\det A_i(\vec{e}_j)}{\det A}.$$

We have to do this n^2 times to compute all the elements of A^{-1}. Notice that $\det A_i(\vec{e}_j) = C_{ji}$ where C is the cofactor matrix. Notice that the indices on C are reversed, so the matrix with elements $\det A_i(\vec{e}_j)$ is the transpose of the cofactor matrix, also called the **adjugate** matrix. So a formula for the inverse of a matrix A is

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{C^T}{\det A}.$$

If the adjugate of A, or the transpose of its cofactor matrix, is known, this gives us an additional way of calculating $\det A$. Notice that we can rearrange the equation above to get

$$\det A \cdot A^{-1} = C^T,$$

then multiply by A from the left to get

$$\det A \cdot I = C^T A,$$

so multiplying $C^T A$ gives the identity matrix with the determinant of A on the diagonal instead of ones.

3.2 Area and Volume

Theorem: The columns of a 2×2 matrix are a pair of vectors that define a parallelogram in $\mathbb{R}^2$. The absolute value of the determinant of such a matrix is the area of that parallelogram. Similarly, the columns of a 3×3 matrix are three vectors that define a parallelepiped in $\mathbb{R}^3$. The absolute value of the determinant of such a matrix is the volume of that parallelepiped.

Since $\det A = \det A^T$, we could also say that the row vectors (instead of the column vectors) define a parallelogram or parallelepiped with area $|\det A|$.

Consider the simple case of

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

A defines a parallelogram that is actually a rectangle with side lengths a and d and area $\det A = ad$. We know that any matrix can be turned into a diagonal matrix (which defines a rectangle) by applying row operations. It turns out that we can turn any parallelogram defined by a matrix A into a rectangle with the same area by performing row operations on A that don't change $|\det A|$.

If the two vectors in a 2×2 matrix are linearly dependent, then the determinant will be zero. The vectors of the parallelogram are the same vector, so this is like calculating the area of a line segment—it is zero.

Example 3.2.1

Calculate the area of the parallelogram with vertices at $(-2, -1)$, $(2, 4)$, $(4, 1)$, and $(8, 6)$.

A matrix represents a parallelogram with one of the vertices at the origin. The parallelogram noted above does not have a vertex at the origin, but we can move it there by adding 2 to every first coordinate and 1 to every second coordinate to get new vertices at $(0, 0)$, $(4, 5)$, $(6, 2)$, and $(10, 7)$. Then the parallelogram is defined by the two sides (i.e. vectors) adjacent to the origin, and the matrix is

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 2 \end{bmatrix}.$$

The area of the parallelogram is $|\det A| = |8 - 30| = 22$.

3.3 Linear Transformations

Consider the set of points $\{S\}$ making up a parallelogram. We can calculate the area of this parallelogram by taking the absolute value of the determinant of the matrix B defining that parallelogram. If we transform this parallelogram into a new parallelogram by applying a linear transformation $T(\vec{x}) = A\vec{x}$ to the points $\{S\}$, how can we find the area of the new parallelogram?

To find the new shape after performing a linear transformation, it is sufficient to find the new boundary by applying the transformation to the original boundary. If the old parallelogram is defined by the vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_{\boldsymbol{2}}$ such that $B = \begin{bmatrix} \vec{\boldsymbol{b}}_1 & \vec{\boldsymbol{b}}_2 \end{bmatrix}$, then the result of the transformation $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is $T(B) = \begin{bmatrix} T(\vec{\boldsymbol{b}}_1) & T(\vec{\boldsymbol{b}}_2) \end{bmatrix} = AB$. Then, because $\det AB = \det A \cdot \det B$, we have that

$$\begin{array}{c}\text{area of new}\\ \text{parallelogram}\end{array} = |\det A| \times \begin{array}{c}\text{area of old}\\ \text{parallelogram}\end{array}.$$

Similarly, in $\mathbb{R}^3$,

$$\begin{array}{c}\text{volume of new}\\ \text{parallelepiped}\end{array} = |\det A| \times \begin{array}{c}\text{volume of old}\\ \text{parallelepiped}\end{array}.$$

Example 3.3.1

If $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

and $\{S\}$ is the set of points in a parallelogram with a vertex at the origin and adjacent vertices at $(1, 3)$ and $(3, 1)$, find the area of the new parallelogram formed by applying the transformation T to $\{S\}$.

We know that the area of the old parallelogram is the absolute value of the determinant of

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},$$

which is 8. The determinant of A is -2, so the area of the new parallelogram is

$$\text{area} = |-2| \times 8 = 16.$$

We can use the same process to calculate the area of a transformed region if the original region is an arbitrary shape. If we have a linear transformation $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$, then applying that transformation to a region S gives us $T(S) = \det A \times \text{area of } S$. In other words, S can be any finite region. To justify this, we can imagine chopping the arbitrary region into many parallelograms and then calculating the area of all the original parallelograms as well as all the transformed parallelograms.

$$\begin{array}{c}\text{area of new}\\ \text{region}\end{array} = |\det A| \times \begin{array}{c}\text{area of old}\\ \text{region}\end{array}.$$

Similarly, in $\mathbb{R}^3$,

$$\begin{array}{c}\text{volume of new}\\ \text{region}\end{array} = |\det A| \times \begin{array}{c}\text{volume of old}\\ \text{region}\end{array}.$$

For example, to find the area of an ellipse with radii 2 and 3, we can find a transformation that transforms the unit circle into the ellipse. We know that the region bounded by the ellipse is

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} \leq 1,$$

and the region bounded by the unit circle is

$$x^2 + y^2 \leq 1.$$

The matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

is the transformation we're looking for since A applied to any vector in the unit disk gives a vector in the ellipse. We know that the area of the unit circle is π so the area of the ellipse is $\pi \times |\det A| = 6\pi$. This example can be generalized to any ellipse by replacing 2 and 3 with arbitrary constants.

3.4 Summary: Determinants

Theorem: If $\det A = 0$, then A is not invertible. If $\det A \neq 0$, then A is invertible. This is an important theorem that needs to be memorized.

For a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc.$$

Using Laplace expansion, the determinant of an $n \times n$ matrix A expanded along the ith row can be computed as

$$\det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} = \sum_{j=1}^{n} A_{ij}C_{ij}.$$

The **minor** M_{ij} of A is the determinant of the submatrix obtained by deleting row i and column j from matrix A.

A **cofactor** C_{ij} is a signed minor obtained by multiplying the minor M_{ij} by negative one if the sum of the indices $i + j$ is odd.

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is a minor.

Properties of determinants:

1. Interchanging or swapping any two rows or columns causes the determinant to change sign.
2. The determinant is unchanged if a multiple of a row (or column) is added to a row (or column).
3. Multiplying a row or a column by a constant changes the determinant by the same factor. If the entire matrix is multiplied by a factor than the determinant is multiplied by the factor raised to the power of the number of rows (or columns). For an $n \times n$ matrix A

$$\text{If } \det(A) = d, \text{ then } \det(\alpha A) = \alpha^n d.$$

4. The determinant is zero if the columns (or rows) of a matrix are linearly dependent.
5. The determinant of the inverse of a matrix is the inverse of the determinant of the matrix

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

6. The determinant of a matrix is equal to the determinant of the transpose of the matrix

$$\det(A^T) = \det(A).$$

7. The determinant of a product of matrices is the product of the determinants of the matrices

$$\det(AB) = \det(A) \cdot \det(B).$$

8. The determinant of a matrix in triangular form is the product of the diagonal elements.

A second way to calculate determinants is to transform the matrix into triangular form then the determinant of that matrix will be the product of the diagonals. In order to calculate the determinant of the original matrix, you have to adjust this determinant based on the row operations you performed. The first three properties listed above apply to row operations.

For an $n \times n$ matrix A and $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, **Cramer's rule** states that

$$x_i = \frac{\det A_i(\vec{\boldsymbol{b}})}{\det A} = \frac{\left|\vec{\boldsymbol{a}}_1 \cdots \vec{\boldsymbol{b}} \cdots \vec{\boldsymbol{a}}_n\right|}{|A|},$$

where x_i is the ith component of $\vec{\boldsymbol{x}}$ and $A_i(\vec{\boldsymbol{b}})$ is the matrix A with the ith column replaced by $\vec{\boldsymbol{b}}$.

A formula for the inverse of a matrix A derived from Cramer's rule is

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{C^T}{\det A}.$$

The columns of a 2×2 matrix A define a parallelogram with area $|\det A|$. The columns of a 3×3 matrix A define a parallelepiped with volume $|\det A|$. If given the vertices and none of them are at the origin, the parallelogram must be translated so that one vertex is at the origin before it can be represented by a matrix.

If we have a region (i.e. set of points) S, and we transform that region into a new region $T(S)$ by applying the linear transformation $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$, then

$$\begin{matrix}\text{area/volume} \\ \text{of } T(S)\end{matrix} \quad = \quad |\det A| \quad \times \quad \begin{matrix}\text{area/volume} \\ \text{of } S\end{matrix} \; .$$

Chapter 4

Vector Spaces

A vector space V is a set of elements called "vectors" with the following properties where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and c and d are scalars:

1. If $\vec{u}$ and $\vec{v}$ are in V then $c\vec{u} + d\vec{v}$ is also in V. That is, any linear combinations of the vectors in a vector space must also be vectors in the vector space.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. That is, vector addition is commutative.
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$. That is, vector addition is associative.
4. The zero vector $\vec{0}$ is in V and $\vec{u} + \vec{0} = \vec{u}$.
5. If $\vec{u}$ is in V, then so is $-\vec{u}$ and $\vec{u} + (-\vec{u}) = \vec{0}$.
6. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.
7. $(c + d)\vec{u} = c\vec{u} + d\vec{v}$.
8. $(cd)\vec{u} = c(d\vec{u})$.

For example, $\mathbb{R}^n$ is the most commonly used vector space. $\mathbb{R}^2$ contains all two-dimensional vectors with real components and $\mathbb{R}^3$ contains all three-dimensional vectors with real components.

P^n is the set of all polynomials of degree n or less with real coefficients. P^4 satisfies all the properties of a vector space, so it is a vector space. For example, possible vectors in P^4 include $x^4 + 3x^3 + x - 1$, x, and $5x^4 + 3$.

The ideas of linear independence, span, and basis exist in P^n just like they do in $\mathbb{R}^n$.

Example 4.0.1

Is the set of vectors $\{x, x^3 + 3x - 4, \frac{2}{3}x^3 + \frac{5}{3}x + \frac{8}{3}\}$ linearly independent?

Recall the meaning of linear independence. If the vectors are linearly independent, then no linear combination of them can produce the zero vector. In our case, if we name the vectors as

$$\begin{aligned}\vec{v}_1 &= x\\ \vec{v}_2 &= x^3 + 3x - 4\\ \vec{v}_3 &= \frac{2}{3}x^3 + \frac{5}{3}x + \frac{8}{3}.\end{aligned}$$

Since

$$\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3 = \vec{0},$$

the vectors are not linearly independent.

Since a linear combination of monomials cannot produce a polynomial of a higher degree, a basis for P^n is

$$\{1, x, x^2, x^3, \ldots, x^n\}.$$

These vectors are linearly independent and they span P^n. Notice that P^n has a dimension of $n + 1$.

Choosing coefficients for a polynomial is the same as taking linear combinations of the basis vectors. To write a polynomial as a vector using the basis vectors given above, we write for example,

$$x^4 - 3x^2 + 4 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}.$$

The magnitude of a vector in P^4 is defined differently than the magnitude of a vector in $\mathbb{R}^n$. In $\mathbb{R}^n$, the size of a vector is the square root of the dot product of the vector with itself. In P^n, the magnitude of the vector is defined in terms of an integral of the polynomial squared. The basis vectors $\{1, x, x^2, x^3, \ldots, x^n\}$ are not orthogonal when magnitude is defined this way.

Chapter 5

Eigenvalues and Eigenvectors

An eigenvalue is a special value associated with a matrix and a vector called an eigenvector in that the matrix times the eigenvector results in the eigenvector times a scalar called the eigenvalue. Eigenvalues sort of characterize the "size" of a matrix.

If and only if you have a matrix equation of the form

$$A\vec{x} = \lambda\vec{x},$$

that is, if the result of $A\vec{x}$ is proportional to $\vec{x}$, then $\vec{x}$ is an **eigenvector** of A and λ is an **eigenvalue** of A. Specifically, $\vec{x}$ is the eigenvector corresponding to λ.

Note: 0 can be an eigenvalue, but by definition, $\vec{\mathbf{0}}$ is never an eigenvector.

To check if a given vector is an eigenvector of a given matrix, just multiply the two and see if the result is proportional to the given vector.

Example 5.0.1

Check if

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ -3 & -3 & 2 \end{bmatrix}.$$

We want to know if $A\vec{x} = \lambda\vec{x}$, so performing the multiplication, we get

$$A\vec{v} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\vec{v}.$$

So $\lambda = 2$ is an eigenvalue of A with the corresponding eigenvector $\vec{v}$.

To check if a given number λ is an eigenvalue of a given matrix A, we have to find non trivial solutions to

$$(A - \lambda I)\vec{x} = \vec{\mathbf{0}}.$$

Notice that this is just a rearrangement of the definition of an eigenvalue $A\vec{x} = \lambda\vec{x}$. Since $\vec{0}$ is not an eigenvector, $\vec{x}$ is an eigenvector if the columns of $A - \lambda I$ are dependent. That is, the columns do not span $\mathbb{R}^n$. So in order for λ to be an eigenvalue, we must be able to find non-trivial solutions $\vec{x}$ for $(A - \lambda I)\vec{x} = \vec{0}$.

Example 5.0.2

Check if 5 is an eigenvalue of

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 4 & -2 \\ -3 & -3 & 2 \end{bmatrix},$$

and if it is, find the corresponding eigenvector.

In order for λ to be an eigenvalue, we must be able to find non-trivial solutions $\vec{x}$ for $(A - \lambda I)\vec{x} = \vec{0}$. If $\lambda = 5$, then the matrix $A - \lambda I$ is

$$\begin{bmatrix} 1-5 & -2 & -1 \\ 1 & 4-5 & -2 \\ -3 & -3 & 2-5 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -1 \\ 1 & -1 & -2 \\ -3 & -3 & -3 \end{bmatrix}.$$

To find if there are non-trivial solutions to $\vec{x}$ for $(A - \lambda I)\vec{x} = \vec{0}$, we use the augmented matrix and transform it into reduced echelon form.

$$\left[\begin{array}{ccc|c} -4 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ -3 & -3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly, there are nontrivial solutions since there are infinite solutions. The general solution is

$$\vec{x} = \begin{bmatrix} \frac{1}{2}x_3 \\ -\frac{3}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}.$$

Since x_3 can take on any values, there are infinite eigenvectors associated with this eigenvalue. The eigenvectors form a line. When giving an eigenvector, we give a convenient one with the understanding that any scaled version of this eigenvector is also an eigenvector. In our case, we find a convenient one by letting $x_3 = 2$. So the eigenvector associated with $\lambda = 5$ is

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

If

$$(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}},$$

for a given eigenvalue λ, then $\vec{\boldsymbol{x}}$ is in $\text{Nul}(A - \lambda I)$. This subspace consisting of the zero vector and all the eigenvectors corresponding to the eigenvalue λ is a subspace called the **eigenspace**. Remember that to form a basis, the vectors must be linearly independent, and they must span the solution space.

Example 5.0.3

Find a basis for the eigenspace of

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 4 & -2 \\ -3 & -3 & 2 \end{bmatrix},$$

corresponding to the eigenvalue 5.

From the previous example, we know that $(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ has the general solution

$$\vec{\boldsymbol{x}} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix},$$

so the basis of the eigenspace is just

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} \right\}.$$

Since this vector spans the solution space it forms the basis of the eigenspace of A associated with the eigenvalue 5.

Theorem: The eigenvalues of a square triangular matrix is the diagonal elements. So the eigenvalues of any triangular matrix can be read directly from the matrix. If a matrix is not triangular, putting it into triangular form will not give you the eigenvalues on the diagonal.

Theorem: If the set of eigenvectors $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n$ of a matrix correspond to *distinct* eigenvalues of the matrix, then the eigenvectors $\{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$ are linearly independent. In other words, if none of the eigenvalues of an $n \times n$ matrix are repeated, then its eigenvectors form a basis for $\mathbb{R}^n$, and any vector in $\mathbb{R}^n$ can be uniquely represented as a linear combination of $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n$. If $\vec{\boldsymbol{x}}$ is in $\mathbb{R}^n$, then

$$\vec{\boldsymbol{x}} = c_1\vec{\boldsymbol{v}}_1 + \cdots + c_n\vec{\boldsymbol{v}}_n.$$

This gives us a new way to think about matrix multiplication

$$\begin{aligned} A\vec{\boldsymbol{x}} &= A(c_1\vec{\boldsymbol{v}}_1 + \cdots + c_n\vec{\boldsymbol{v}}_n) \\ &= c_1\lambda_1\vec{\boldsymbol{v}}_1 + \cdots + c_n\lambda_n\vec{\boldsymbol{v}}_n. \end{aligned}$$

Applying A repeatedly gives us

$$A^p\vec{\boldsymbol{x}} = c_1\lambda_1^p\vec{\boldsymbol{v}}_1 + \cdots + c_n\lambda_n^p\vec{\boldsymbol{v}}_n,$$

which gives us a much easier way of calculating a power of A than to multiply A by itself repeatedly.

Recall that if λ is an eigenvalue of A and the correponding eigenvector is $\vec{\boldsymbol{x}}$, then

$$(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}},$$

has non-trivial solutions. By the invertible matrix theorem, we know that the equation above has non-trivial solutions (i.e. infinite solutions) only when $(A-\lambda I)$ is not invertible, and when it is not invertible, we know that $\det(A - \lambda I) = 0$. So to find the eigenvalues of A, we find all values λ such that

$$\det(A - \lambda I) = 0.$$

When we take the determinant of $(A - \lambda I)$, expand it and set it equal to zero, we get what is called the **characteristic equation** of A. After finding each eigenvalue, the corresponding eigenvector can be found by plugging it back into $(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ and solving for the nontrivial solution.

Example 5.0.4

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We know that $\det(A - \lambda I) = 0$ and doing the subtraction and taking the determinant, gives us

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \\ &= (-1-\lambda)(4-\lambda) - 6 = 0 \\ &= \lambda^2 - 3\lambda - 10 = 0 \\ &= (\lambda - 5)(\lambda + 2) = 0. \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -2$. To determine the eigenvector corresponding to $\lambda_1 = 5$, we plug this back into $(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ and solve it to get

$$\left[\begin{array}{cc|c} -1-5 & 2 & 0 \\ 3 & 4-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -6 & 2 & 0 \\ 3 & -1 & 0 \end{array}\right],$$

whose RREF is

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This gives us $x_1 = \frac{1}{3}x_2$, so the general solution is

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Since there are infinite eigenvectors corresponding to an eigenvalue (i.e. they form a line), we pick a convenient one, in this case

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

which we call *the* eigenvector corresponding to the eigenvalue 5.

We go through the same process to find the eigenvector corresponding to $\lambda = -2$. Plugging it into $(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ and solving gives us

$$\left[\begin{array}{cc|c} -1+2 & 2 & 0 \\ 3 & 4+2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right],$$

whose RREF is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This gives us $x_1 = -2x_2$, so the general solution is

$$\vec{\boldsymbol{x}} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

so a convenient eigenvector corresponding to $\lambda = -2$ is

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

An eigenvalue that is repeated s times in a matrix, is said to have **multiplicity** s. For example, if the eigenvalues of a matrix are 1, 1, 2, then the eigenvalue 1 has multiplicity 2.

Any $n \times n$ matrix has an nth degree characteristic polynomial, and therefore has n eigenvalues if you account for multiplicities.

So a 3×3 matrix will have 3 eigenvalues. Its characteristic equation will be cubic, so finding the eigenvalues is not necessarily easy (or possible) for a matrix larger than 2×2.

Theorem: A matrix is invertible if and only if its determinant is nonzero *and* it does not have a zero eigenvalue. This follows from the fact that if $\lambda = 0$, then $\det(A - \lambda I) = \det A = 0$, which implies that A is not invertible.

Two $n \times n$ matrices A and B are **similar** if and only if they can be written as $P^{-1}AP = B$, which is equivalent to

$$A = PBP^{-1}.$$

Geometrically, similar matrices represent the same linear transformation expressed in different coordinate systems.

Theorem: If two matrices are similar, they have the same eigenvalues. The converse is not true. Two matrices with the same eigenvalues are not necessarily similar.

The proof of this theorem is fairly simple.

$$\begin{aligned}
B - \lambda I &= P^{-1}AP - \lambda I \\
&= P^{-1}AP - \lambda P^{-1}P \\
&= P^{-1}(A - \lambda I)P \\
\det(B - \lambda I) &= \det(P^{-1}(A - \lambda I)P) \\
&= \det P^{-1} \det(A - \lambda I) \det P \\
&= \det(A - \lambda I) \det P^{-1} \det P \\
&= \det(A - \lambda I) \det(P^{-1}P) \\
&= \det(A - \lambda I) \det I \\
&= \det(A - \lambda I).
\end{aligned}$$

Since $\det(B - \lambda I) = \det(A - \lambda I)$, they must have the same characteristic equations and therefore the same eigenvalues.

A square matrix is **diagonalizable** if it is similar to a diagonal matrix. That is, a matrix A is diagonalizable, if it can be written as

$$A = PDP^{-1},$$

where D is a diagonal matrix.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors or if (but not only if) it has n distinct eigenvalues. If it is diagonalizable, it can be written in the form

$$A = PDP^{-1},$$

where P is the matrix whose columns are the eigenvectors of A and D is a diagonal matrix whose elements are the eigenvalues of A corresponding to the eigenvectors in P.

The above theorem is proved by considering AP where $P = [\vec{\boldsymbol{v}}_1 \cdots \vec{\boldsymbol{v}}_n]$, where $\vec{\boldsymbol{v}}_1 \cdots \vec{\boldsymbol{v}}_n$ are the eigenvectors of A. Then

$$\begin{aligned}
AP &= A\,[\vec{\boldsymbol{v}}_1 \cdots \vec{\boldsymbol{v}}_n] \\
&= [A\vec{\boldsymbol{v}}_1 \cdots A\vec{\boldsymbol{v}}_n] \\
&= [\lambda_1\vec{\boldsymbol{v}}_1 \cdots \lambda_n\vec{\boldsymbol{v}}_n] \\
&= PD,
\end{aligned}$$

where D is the diagonal matrix with the eigenvalues of A on the diagonal. Multiplying from the left by P^{-1} gives us $A = PDP^{-1}$. Notice that P must be invertible, and it is if the eigenvectors of A are linearly independent. Since the eigenvectors of A are linearly independent, they form an **eigenvector basis** for $\mathbb{R}^n$.

The matrix A does not have to be invertible to be diagonalizable. Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is diagonalizable since it is already diagonal. Then P is just the identity matrix. However, the column vectors of this matrix are not linearly independent, so it is not invertible.

Example 5.0.5

Determine if the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix},$$

is diagonalizable.

Since the matrix is triangular, the eigenvalues are just the diagonal elements—1, 0, and 5. Notice also that the determinant, which is the product of the eigenvalues, is 0, so the matrix is not invertible. However, since there are 3 distinct eigenvalues, the matrix is diagonalizable.

If an $n \times n$ matrix doesn't have n distinct eigenvalues, it may or may not be diagonalizable. It depends on the dimension of the eigenspace (i.e. the span of the eigenvectors). In general, if the dimension of the eigenspace is equal to n, then the matrix is diagonalizable. In other words, the eigenvectors of A have to form a basis for $\mathbb{R}^n$.

One application of diagonalization is that it makes it much easier to compute a power of a matrix A since

$$A^k = PD^kP^{-1}.$$

It is much easier since a diagonal matrix raised to a power is just the matrix with the diagonal elements raised to the power.

To diagonalize an $n \times n$ matrix A, we follow these steps:

1. Find the eigenvalues of A
2. Find n linearly independent eigenvectors of A
3. We can now construct P and D using what we found in steps 1 and 2

An $n \times n$ matrix is diagonalizable if and only if the sum of the dimensions of the eigenspaces is n.

5.1 Eigenvectors and Linear Transformations

Recall that the standard matrix A for a linear transformation $T(\vec{x}) = A\vec{x}$ where T maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ is

$$A = [T(\vec{e}_1) \cdots T(\vec{e}_m)],$$

where $\vec{e}_i$ are the standard basis vectors. What if we don't have the standard basis, but some basis $B = \{\vec{b}_1, \ldots, \vec{b}_m\}$ for $\mathbb{R}^m$ and $C = \{\vec{c}_1, \ldots, \vec{c}_n\}$ for $\mathbb{R}^n$? Then, since $\vec{x}$ is in $\mathbb{R}^m$, the B-coordinate vector of $\vec{x}$ is $[\vec{x}]_B$. Similarly, since $T(\vec{x})$ is in $\mathbb{R}^n$, the C-coordinate vector of $T(\vec{x})$ is $[T(\vec{x})]_C$. Now, instead of $T(\vec{x}) = A\vec{x}$, we have

$$[T(\vec{x})]_C = M\,[\vec{x}]_B\,.$$

We say that M is the "B matrix of T" rather than the *standard* matrix of T since the standard matrix refers to the matrix involved when the basis is the standard basis.

Example 5.1.1

T is a linear transformation that maps from R^2 to R^3, where $\{\vec{b}_1, \vec{b}_2\}$ is a basis for R^2 and $\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$. If $T(\vec{b}_1) = \vec{c}_1 + 2\vec{c}_2 + 3\vec{c}_3$, and $T(\vec{b}_2) = 2\vec{c}_1 + 4\vec{c}_2 + 6\vec{c}_3$, find the B matrix of T. That is, find the matrix M such that

$$[T(\vec{x})]_C = M\,[\vec{x}]_B\,.$$

We know that

$$[\vec{\boldsymbol{x}}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix},$$

where k_i are some constants. That is, we know that $\vec{\boldsymbol{x}} = k_1\vec{\boldsymbol{b}}_1 + k_2\vec{\boldsymbol{b}}_2$. Since T is a linear transformation, its action on $\vec{\boldsymbol{x}}$ is

$$\begin{aligned} T(\vec{\boldsymbol{x}}) &= T(k_1\vec{\boldsymbol{b}}_1 + k_2\vec{\boldsymbol{b}}_2) \\ &= T(k_1\vec{\boldsymbol{b}}_1) + T(k_2\vec{\boldsymbol{b}}_2) \\ &= k_1T(\vec{\boldsymbol{b}}_1) + k_2T(\vec{\boldsymbol{b}}_2). \end{aligned}$$

Plugging in what we know, gives us

$$\begin{aligned} T(\vec{\boldsymbol{x}}) &= k_1T(\vec{\boldsymbol{b}}_1) + k_2T(\vec{\boldsymbol{b}}_2) \\ &= k_1(\vec{\boldsymbol{c}}_1 + 2\vec{\boldsymbol{c}}_2 + 3\vec{\boldsymbol{c}}_3) + k_2(2\vec{\boldsymbol{c}}_1 + 4\vec{\boldsymbol{c}}_2 + 6\vec{\boldsymbol{c}}_3) \\ &= (k_1 + 2k_2)\vec{\boldsymbol{c}}_1 + (2k_1 + 4k_2)\vec{\boldsymbol{c}}_2 \\ &\qquad +(3k_1 + 6k_2)\vec{\boldsymbol{c}}_3. \end{aligned}$$

In other words

$$[T(\vec{\boldsymbol{x}})]_C = \begin{bmatrix} k_1 + 2k_2 \\ 2k_1 + 4k_2 \\ 3k_1 + 6k_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix},$$

so

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Notice that we could have simply plugged in the given coefficients in $T(\vec{\boldsymbol{b}}_1)$ and $T(\vec{\boldsymbol{b}}_2)$ as the columns of M.

As demonstrated in the example above, the matrix M in

$$[T(\vec{\boldsymbol{x}})]_C = M\,[\vec{\boldsymbol{x}}]_B\,,$$

is just

$$M = \left[\left[T(\vec{\boldsymbol{b}}_1)\right]_C \cdots \left[T(\vec{\boldsymbol{b}}_m)\right]_C\right].$$

For the special case that a linear transformation is going from a vector space $\mathbb{R}^n$ to the same vector space, we have that

$$[T(\vec{\boldsymbol{x}})]_B = M\,[\vec{\boldsymbol{x}}]_B\,,$$

where M is the B matrix of T, sometimes denoted $M = [T]_B$.

Theorem: For a linear transformation $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ that maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and A is diagonalizable as $A = PDP^{-1}$, then if B is the basis formed by the eigenvectors of

A, then D is the B matrix of T. This theorem allows us to find a diagonal B matrix of T by using the eigenvectors of A as the basis B.

If you're given A and $B = \{\vec{b}_1, \vec{b}_2, \ldots\}$ and asked to find $[T]_B$, you can use this theorem even if the basis B is not the eigenvectors of A. That is because if A is similar to C, then $A = PCP^{-1}$ and $B = \{\vec{P}_1, \vec{P}_2, \ldots\}$ then C is the B matrix of A. In other words, for $\mathbb{R}^2$, $A = PCP^{-1} = [\vec{b}_1\ \vec{b}_2][T]_B[\vec{b}_1\ \vec{b}_2]^{-1}$ and you can solve for $[T]_B$ as

$$[T]_B = [\vec{b}_1\ \vec{b}_2]^{-1}A[\vec{b}_1\ \vec{b}_2].$$

To go from $\vec{x}$ to $A\vec{x}$, multiply by A. To go from $\vec{x}$ to $[\vec{x}]_B$, multiply by P^-1. To go from $[\vec{x}]_B$ to $[A\vec{x}]_B$, multiply by D. To go from $[A\vec{x}]_B$ to $A\vec{x}$, multiply by P.

5.2 Complex Vectors and Eigenvectors

A complex vector of matrix is just a vector or matrix with complex entries. The complex conjugate of a complex vector $\vec{v}$ is just the vector containing the complex conjugates of the elements of $\vec{v}$. The magnitude of a complex vector is the square root of the dot product of the vector with its complex conjugate. To take the complex conjugate of anything, you just change the sign of every i.

Example 5.2.1

Find the real part, imaginary part, complex conjugate, and magnitude of the vector

$$\vec{z} = \begin{bmatrix} 3+2i \\ i \\ 5 \end{bmatrix}.$$

The real part, imaginary part, and complex conjugate are

$$\Re(\vec{z}) = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \Im(\vec{z}) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

$$\vec{z}^* = \bar{\vec{z}} = \begin{bmatrix} 3-2i \\ -i \\ 5 \end{bmatrix}.$$

The magnitude of $\vec{z}$ is

$$\begin{aligned} ||\vec{z}|| &= \sqrt{\vec{z} \cdot \vec{z}^*} \\ &= \sqrt{(3+2i)(3-2i) + (i)(-i) + (5)(5)} \\ &= \sqrt{39}. \end{aligned}$$

For a vector $\vec{v}$, a matrix A, and a scalar c (all possibly complex),

$$\begin{aligned}(c\vec{v})^* &= c^*\vec{v}^* \\ (cB)^* &= c^*B^* \\ (B\vec{v})^* &= B^*\vec{v}^*.\end{aligned}$$

If B is real, then $B^* = B$.

We know that an $n \times n$ matrix A has n eigenvalues. But consider a 2×2 rotation matrix that when multiplied to a vector $\vec{x}$, rotates the vector by a certain angle. If you repeatedly apply this matrix to a given vector, the vector is rotated about the origin, and eventually, it will return to the same position. In other words, $A^n\vec{x}$ spins the vector and it stays on a circle. Since the length of the vector is never changed, then no nonzero vector is ever mapped to a scaled version of itself, and so $A\vec{x} = \lambda\vec{x}$ for no real eigenvalues. Therefore, the two eigenvectors must be complex. So even if our matrices don't have complex entries, they can have complex eigenvalues and eigenvectors.

For a real matrix A, a complex eigenvalue λ, and a complex eigenvector $\vec{v}$, we have that $A\vec{v} = \lambda\vec{v}$. Taking the complex conjugate of both sides, we have that $(A\vec{v})^* = A\vec{v}^* = \lambda^*\vec{v}^*$. This tells us that if λ is a complex eigenvector of A, then so is λ^*. In other words, complex eigenvalues and eigenvectors come in conjugate pairs.

Example 5.2.2

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Using $\det(A - \lambda I) = 0$ we find that the characteristic equation is

$$\lambda^2 - 4\lambda + 5 = 0,$$

which gives us the complex eigenvalues

$$\lambda = 2 \pm i.$$

To find the eigenvectors, we use $(A - \lambda I)\vec{x} = \vec{0}$ as usual, but we only have to find one eigenvector since we know that the other one will be its complex conjugate. We find that

$$(A - \lambda I) = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix},$$

so the augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right].$$

Doing row operations is difficult with complex numbers, but we can find the eigenvectors without row operations. Since the determinant is zero, we know that the rows are not linearly independent, so we can just pick any row and it will give us the same result as picking any other row. In our case, we pick the second row to find that $-x_1 + (1-i)x_2 = 0$. So

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1-i)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix},$$

so for our first eigenvector we choose $x_2 = 1$. Since we know the second eigenvector is the complex conjugate of the first, our eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}, \qquad \vec{x}_2 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

5.3 Dynamical Systems

Recall the difference equation

$$\vec{x}_{k+1} = A\vec{x}_k.$$

Plotting these, $\vec{x}_0$, $\vec{x}_1$, $\vec{x}_2$, and so on, gives us a **trajectory**. For a 2×2 matrix, each of these vectors specifies a point in $\mathbb{R}^2$, so given a matrix A and an initial point $\vec{x}_0$, we can plot the solution of the difference equation as a trajectory.

To solve this kind of problem, we typically find the eigenvalues λ_1, λ_2, and the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A since it greatly simplifies repeated matrix multiplication as

$$\vec{x}_k = A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2.$$

To calculate powers of matrices, we utilize the eigenvalues and eigenvectors of the matrix since for linearly independent eigenvectors, Applying A repeatedly gives us

$$A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + \cdots + c_n\lambda_n^k\vec{v}_n.$$

Another way of thinking about this is in terms of diagonalization since

$$A^k = PD^kP^{-1}.$$

The general rule for plotting the trajectories caused by 2×2 diagonal matrices

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

is

- If $|d_1| < 1$ then the trajectory is attracted to the origin in the x_1 direction.
- If $|d_1| > 1$ then the trajectory is repelled from the origin in the x_1 direction.
- If $|d_1| = 1$ then the value of x_1 is constant.

The result of this is

- An **attractor** or sink if all eigenvalues are less than 1
- A **repeller** or source if all eigenvalues are larger than 1
- A **saddle** if they are mixed

For a non-diagonal matrix, the process of plotting the trajectory is similar, but finding the eigenvalues and eigenvectors is not as easy as with diagonal matrices where the eigenvalues can be read directly from the matrix. For non-diagonal matrices, we find the eigenvalues and eigenvectors and construct the same kind of equation for $\vec{x}_k$. However, for non-diagonal matrices, the symmetry will be about the eigenvectors instead of about the x_1 and x_2 axes. So draw the eigenvectors onto the graph, and choose a point on an eigenvector for $\vec{x}_0$ to determine whether the trajectory is going toward or away from the origin for that direction. Then do the same with the other eigenvector. From the behavior in these two directions, the overall behavior can be drawn in using curved lines with arrows to indicate the direction of the trajectories.

Example 5.3.1

Plot the general and specific solution of $\vec{\boldsymbol{x}}_{k+1} = A\vec{\boldsymbol{x}}_k$ if

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \qquad \vec{\boldsymbol{x}}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Recall that for a diagonal matrix, the eigenvalues are the diagonal elements and the eigenvectors are just the standard basis vectors, so

$$\vec{\boldsymbol{x}}_k = c_1(0.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.7)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

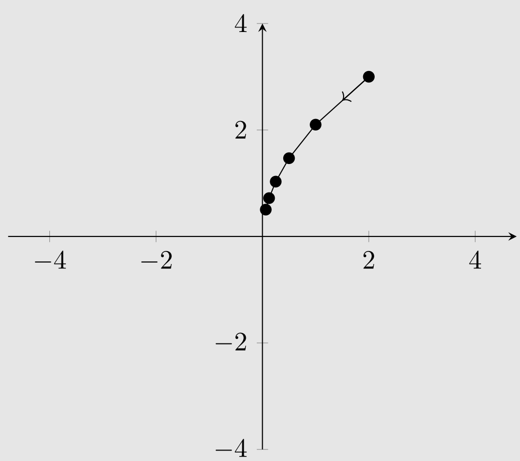
For the specific case, we are given a value for $\vec{\boldsymbol{x}}_0$ so we plug in $k = 0$ into the equation above to get

$$\vec{\boldsymbol{x}}_0 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

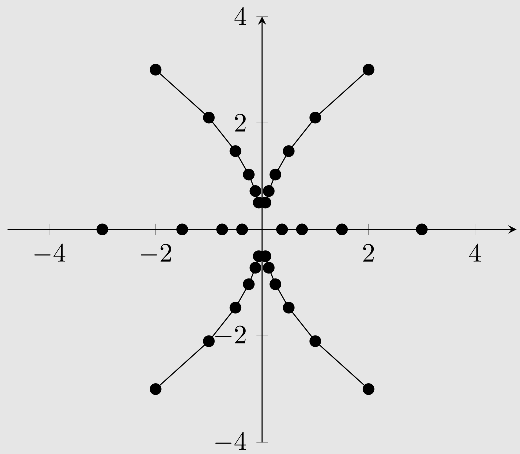
so $c_1 = 2$ and $c_2 = 3$, so our specific solution is

$$\vec{\boldsymbol{x}}_k = 2(0.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3(0.7)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By plugging in different values for k, we can plot a trajectory for the specific solution.



Notice that the trajectory curves inward toward the center. This occurs because both eigenvalues are less than 1, so when they are raised to increasing powers, the both shrink. For the general solution, we refer back to the equation containing c_1 and c_2, which can take on any values specified by an arbitrary initial condition. For example, we let the it start on the horizontal axis, then $c_2 = 0$, and we can see that the trajectory stays on the horiztonal axis and moves toward the origin. We have the same case with starting on the vertical axis. In fact, every trajectory shrinks toward the origin, so this is called an "attractor". The specific solution show above is just one possible trajectory. The general solution looks more like this:



If the eigenvalues and eigenvectors are complex, then we can't diagonalize the matrix as $A^k = PD^kP^{-1}$, but we can do something similar.

For a 2×2 matrix A with complex eigenvalues $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ and eigenvectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_1^*$, we can write

$$A = PCP^{-1},$$

where

$$P = \left[\Re(\vec{\boldsymbol{v}}_1) \quad \Im(\vec{\boldsymbol{v}}_1)\right], \qquad C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

For a matrix with complex eigenvalues, a power of the matrix can be calculated as

$$A^k = PC^kP^{-1}.$$

If the magnitude of the eigenvalues is 1, then $a^2 + b^2 = 1$, which means C is a rotation matrix. Then $C^kP^{-1}\vec{\boldsymbol{x}}$ puts everything on a circle, and then left-multiplication by P contorts the circle. So $A^k\vec{\boldsymbol{x}}$, if C is a rotation matrix, results in a contorted circle if plotted on $\mathbb{R}^2$. This contorted circle is called a **trajectory**.

We can also write

$$C = r\begin{bmatrix} \frac{a}{r} & \frac{b}{r} \\ -\frac{b}{r} & \frac{a}{r} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

where $r = |\lambda| = \sqrt{a^2 + b^2}$. Now, the matrix is definitely a rotation matrix, and it is scaled by a factor r. Then

- If $r = 1$, C is a rotation matrix. If C is a rotation matrix, then the trajectory of $A^k\vec{\boldsymbol{x}}$ will be a closed curve. The rotation matrix C^k puts the trajectory on a circle, but then PC^kP^{-1} contorts the circle.
- If $r > 1$, then C is a rotation matrix scaled by r. C^k still spins the vectors, but it also makes them larger, so in this case, the trajectory of $A^k\vec{\boldsymbol{x}}$ spirals outward.

- If $r < 1$, then C is again a scaled rotation matrix, but this time the trajectory spirals inward. For example, a damped oscillating system, since the oscillations are getting smaller and smaller, could be represented by some $A^k\vec{\boldsymbol{x}}$ where $r < 1$.

So in order to plot the trajectory of $A^k\vec{\boldsymbol{x}}$, where A is a 2×2 matrix with complex eigenvalues, then all we need to know is the value of r in order to determine if the trajectory is an inward spiral or an outward spiral. So all we need to do is find one eigenvalue $\lambda = a + bi$ of the matrix then calculate $r = \sqrt{a^2 + b^2}$. However, there is one ambiguity, and that's whether the trajectory spirals clockwise or counterclockwise. To determine the direction of the spiral, we simply multiply A by a convenient vector $\vec{\boldsymbol{x}}_0$ and then the arrow on $\mathbb{R}^2$ drawn from $\vec{\boldsymbol{x}}_0$ to $A\vec{\boldsymbol{x}}_0$ tells us whether the spiral is clockwise or counterclockwise.

Example 5.3.2

Determine the behavior of A^k on a vector $\vec{\boldsymbol{x}}$ if

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

In a previous example, we found that the eigenvalues of this matrix are $\lambda = 2 \pm i$, so we have that

$$r = \sqrt{2^2 + 1^2} = \sqrt{3}.$$

Since $r > 1$, the trajectory of $A^k\vec{\boldsymbol{x}}$ is an outward spiral centered on the origin. Choosing a convenient vector on the x_1 axis and multiplying it by A gives us the two points

$$\vec{\boldsymbol{x}}_0 = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}, \qquad A\vec{\boldsymbol{x}}_0 = \begin{bmatrix} 1 \\ \\ -1 \end{bmatrix}.$$

Plotting these two points and drawing an arrow from the first to the second, illustrates that the outward spiral is in the clockwise direction. To summarise the general trajectory of $A^k\vec{\boldsymbol{x}}$, we say that for any given initial conditions $\vec{\boldsymbol{x}}_0$, the trajectory of $A^k\vec{\boldsymbol{x}}$ will be a counterclockwise spiral away from the origin.

For a 2×2 matrix A

1. If both eigenvalues are real, then $A = PDP^{-1}$ where P contains the eigenvectors as column vectors, and D is the diagonal matrix containing the eigenvalues.
2. If both eigenvalues are complex, then $A = PCP^{-1}$ where P contains the real and imaginary part of an eigenvector and C contains the real and imaginary part of the eigenvalues.

For a 3×3 matrix, with complex eigenvalues, at least 1 eigenvalue must be real since the matrix has odd dimensions and eigenvalues and eigenvectors come in conjugate pairs.

In summary, when it comes to plotting the trajectory of $A^k\vec{\boldsymbol{x}}$ as k grows from 0 to infinity, we find the eigenvalues and eigenvectors of A, then use the equation $\vec{\boldsymbol{x}}_k = A^k\vec{\boldsymbol{x}} = c_1\lambda_1^k\vec{\boldsymbol{v}}_1 + c_2\lambda_2^k\vec{\boldsymbol{v}}_2$. Then

- If the eigenvalues of A are real, the trajectory is a line.
- If the eigenvalues of A are complex, the trajectory is a rotation of some sort, often a contorted circle or a spiral.
- If the eigenvalues of A are mixed, then the real ones push the trajectory in a certain direction and the complex ones spin the trajectory.

An example of a dynamical system is a predator-prey relationship. Consider the wolf (predator) and deer (prey) relationship. The wolves need to eat the deer in order to survive. However, if they diminish the deer population too much, the wolves will die from lack of food. We can make a predator-prey matrix that tells us the population of each in a given year. For example, if the wolf population in year k is 0.3 times the population in the previous year plus 0.4 times the population of deer in the previous year, then $W_k = 0.3W_{k-1} + 0.4D_{k-1}$. Similarly, the population of deer in year k is $-pW_{k-1} + 1.5D_{k-1}$, where p is a parameter that depends on the level of predation by the wolves. Then the populations of the wolves and deer in year k can be represented by the vector

$$\vec{x}_k = \begin{bmatrix} W_k \\ D_k \end{bmatrix},$$

and the evolution of the system can be represented by the difference equation $\vec{x}_k = A\vec{x}_{k-1}$, where

$$A = \begin{bmatrix} 0.3 & 0.4 \\ -p & 1.5 \end{bmatrix}.$$

What is the long-term behavior of the system if $p = 0.1$?

In this case, the eigenvalues are approximately $\lambda_1 = 1.47$ and $\lambda_2 = 0.334$. The eigenvectors are approximately

$$\vec{v}_1 = \begin{bmatrix} 32 \\ 95 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 100 \\ 9 \end{bmatrix},$$

so their evolution can be modeled by

$$\vec{x}_k = c_1(1.47)^k \begin{bmatrix} 32 \\ 95 \end{bmatrix} + c_2(0.334)^k \begin{bmatrix} 100 \\ 9 \end{bmatrix}.$$

As $k \to \infty$, the second term goes to zero because of the $(0.334)^k$ factor. So in the long-term, their evolution is modeled by

$$\vec{x}_k \approx c_1(1.47)^k \begin{bmatrix} 32 \\ 95 \end{bmatrix},$$

which indicates that the population of both will grow by 47% every year and there will be 32 wolves for every 95 deer.

For what value of p will both populations remain stable in the long-term? In order for the populations to remain stable in the long-term, we need one eigenvalue to be one because $1^k = 1$ and the other eigenvalue to be between zero and one because such a number raised to the power k will decrease to zero as k is increased. Using $\det(A - \lambda I) = 0$ and applying the quadratic formula, we find that

$$\lambda = 0.9 \pm \frac{1}{2}\sqrt{(1.8)^2 - 4(0.4p + 0.45)},$$

so in order for one eigenvalue to be one and the other less than one, we must have that $\frac{1}{2}\sqrt{(1.8)^2 - 4(0.4p + 0.45)} = 0.1$. Solving for p, we find that $p = 0.875$. Then $\lambda_1 = 1$ and $\lambda_2 = 0.8$ and the eigenvectors are approximately

$$\vec{v}_1 = \begin{bmatrix} 50 \\ 87 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 62 \\ 78 \end{bmatrix},$$

so their evolution can be modeled by

$$\vec{\boldsymbol{x}}_k = c_1 \begin{bmatrix} 50 \\ 87 \end{bmatrix} + c_2(0.8)^k \begin{bmatrix} 62 \\ 78 \end{bmatrix}.$$

In the long-term, their evolution is modeled by

$$\vec{\boldsymbol{x}}_k \approx c_1 \begin{bmatrix} 50 \\ 87 \end{bmatrix}.$$

Notice that the population is neither growing nor decreasing, and the ratio of deer to wolves is approximately 1.3.

Now we turn from discrete dynamical systems to continuous dynamical systems which utilize differential equations. Recall that the general solution to the first order differential equation

$$\dot{x} = ax,$$

is

$$x(t) = x_0 e^{at},$$

where $x_0 = x(0)$.

Suppose you have a system of linear differential equations

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots & \qquad \vdots \\ \dot{x}_n &= a_{n1}x_1 + \cdots + a_{nn}x_n. \end{aligned}$$

Then they can be represented by the matrix equation

$$\dot{\vec{x}} = A\vec{\boldsymbol{x}}.$$

If we diagonalize A such that $A = PDP^{-1}$, then

$$\dot{\vec{x}} = PDP^{-1}\vec{\boldsymbol{x}}.$$

Right-multiplying by P^{-1} gives us

$$P^{-1}\dot{\vec{x}} = DP^{-1}\vec{\boldsymbol{x}}.$$

Since P^{-1} contains only numbers, we can pull the derivative operator to the outside on the left.

$$\frac{d}{dt}(P^{-1}\vec{\boldsymbol{x}}) = DP^{-1}\vec{\boldsymbol{x}}.$$

Now, if we make the substitution $\vec{\boldsymbol{y}} = P^{-1}\vec{\boldsymbol{x}}$ on both sides, we get

$$\dot{\vec{y}} = D\vec{\boldsymbol{y}},$$

which means

$$\begin{bmatrix} \dot{y_1} \\ \vdots \\ \dot{y_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Since D is a diagonal matrix of eigenvalues, this differential matrix equation is decoupled, meaning that the derivative of each scalar function $\dot{y}_i$ depends only on the same function y_i. In other words, each scalar equation is a differential equation of the form $\dot{y}_i = \lambda_i y_i$, which we know has the solution $y_i = c_i e^{\lambda_i t}$. So the solution to $\dot{\vec{y}} = D\vec{\boldsymbol{y}}$ is

$$\vec{\boldsymbol{y}} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}.$$

Returning to our substitution $\vec{\boldsymbol{y}} = P^{-1}\vec{\boldsymbol{x}}$, we multiply from the left by P to get $\vec{\boldsymbol{x}} = P\vec{\boldsymbol{y}}$. From diagonalizing A, we know that the columns of P are the eigenvectors of A. Putting it all together, we have that

$$\vec{\boldsymbol{x}} = P\vec{\boldsymbol{y}} = [\vec{\boldsymbol{v}}_1 \cdots \vec{\boldsymbol{v}}_n] \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix},$$

which tells us that the solution to $\dot{\vec{x}} = A\vec{\boldsymbol{x}}$ is

$$\vec{\boldsymbol{x}} = c_1 e^{\lambda_1 t}\vec{\boldsymbol{v}}_1 + \cdots + c_n e^{\lambda_n t}\vec{\boldsymbol{v}}_n,$$

where λ_i are the eigenvalues of A, $\vec{\boldsymbol{v}}_i$ are the eigenvectors of A, and c_i are constants determined by the initial conditions of the problem.

Example 5.3.3

Solve the system of linear differential equations

$$\begin{aligned} \dot{x}(t) &= 6x(t) - 7y(t) \\ \dot{y}(t) &= -3x(t) + 2y(t), \end{aligned}$$

with the initial conditions $x(0) = 2$ and $y(0) = 3$.

We can write this as

$$\dot{\vec{z}} = A\vec{\boldsymbol{z}},$$

where

$$A = \begin{bmatrix} 6 & -7 \\ -3 & 2 \end{bmatrix}, \qquad \vec{\boldsymbol{z}}_0 = \vec{\boldsymbol{z}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 9$ and $\lambda_2 = -1$, and the corresponding eigenvectors are

$$\vec{\boldsymbol{v}}_1 = \begin{bmatrix} -7 \\ 3 \end{bmatrix}, \qquad \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So the general solution is

$$\vec{z}(t) = c_1 e^{9t} \begin{bmatrix} -7 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

or

$$\vec{z} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -7c_1 e^{9t} + c_2 e^{-t} \\ 3c_1 e^{9t} + c_2 e^{-t} \end{bmatrix}.$$

By plotting the eigenvectors on the xy-plane, we can plot the trajectory of the general solution. Note that along $\vec{v}_1$, $c_2 = 0$, and because of e^{9t}, $\vec{z}(t)$ is growing away from the origin. Along $\vec{v}_2$, $c_1 = 0$, and because of e^{-t}, $\vec{z}(t)$ is shrinking toward the origin. So the trajectory forms a saddle.

Setting $t = 0$ and using the given initial conditions, we have that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -7 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solving this vector equation using an augmented matrix or a simple system of equations, we find that $c_1 = \frac{1}{10}$ and $c_2 = \frac{27}{10}$, so our specific solution is

$$\vec{z}(t) = \frac{1}{10} e^{9t} \begin{bmatrix} -7 \\ 3 \end{bmatrix} + \frac{27}{10} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In general, we'll be asked to solve $\dot{\vec{x}} = A\vec{x}$ given a matrix A. To find $\vec{x}(t)$, we just find the eigenvalues and eigenvectors of A and plug them into the equation we found for $\vec{x}(t)$. Then we plot the eigenvectors on the plane and plot the general trajectories of $\vec{x}(t)$ as it varies with time. If we're given initial conditions $\vec{x}(0) = \vec{x}_0$, we solve for the constants in the general solution $\vec{x}(t)$ by setting $t = 0$.

If A is already diagonal, then it is a lot easier to find the solution to $\dot{\vec{x}} = A\vec{x}$ since we can just read the eigenvalues of A from its diagonal elements, and the eigenvectors are just the standard basis vectors.

When plotting the general trajectory, $\lambda > 0$ repels in the direction of $\vec{v}$, and $\lambda < 0$ attracts in the direction of $\vec{v}$.

- If both eigenvalues are positive, it is a repeller or source.
- If both eigenvalues are negative, it is an attractor or sink.
- If one eigenvalue is positive and the other is negative, it is a saddle.

What if the eigenvalues are complex? Recall that complex eigenvalues and eigenvectors come in conjugate pairs. We find the eigenvalues $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ and the corresponding eigenvectors $\vec{v}_1 = \vec{v}$ and $\vec{v}_2 = \vec{v}^*$. Then for $\dot{\vec{x}} = A\vec{x}$ with complex eigenvalues, the general solution has the same form

$$\vec{x} = c_1 e^{(a+bi)t} \begin{bmatrix} c + id \\ f + ig \end{bmatrix} + c_2 e^{(a-bi)t} \begin{bmatrix} c - id \\ f - ig \end{bmatrix}.$$

We have a general solution, however, it is a complex solution, and we want to write it with no imaginary part. We start by defining two functions

$$\begin{aligned}\vec{\boldsymbol{x}}_1 &= e^{(a+bi)t}\begin{bmatrix} c+id \\ f+ig \end{bmatrix}\\ \vec{\boldsymbol{x}}_2 &= e^{(a-bi)t}\begin{bmatrix} c-id \\ f-ig \end{bmatrix}.\end{aligned}$$

Notice that $\vec{\boldsymbol{x}}$ is a linear combination of $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$, and that $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$ are a linearly independent pair that form a basis for $\vec{\boldsymbol{x}}$. We want to find a new basis for $\vec{\boldsymbol{x}}$ that is real. We begin by rewriting $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$ using Euler's formula.

$$\begin{aligned}\vec{\boldsymbol{x}}_1 &= e^{(a+bi)t}\begin{bmatrix} c+id \\ f+ig \end{bmatrix}\\ &= e^{at}e^{ibt}\begin{bmatrix} c+id \\ f+ig \end{bmatrix}\\ &= e^{at}(\cos bt + i\sin bt)\begin{bmatrix} c+id \\ f+ig \end{bmatrix}\\ &= e^{at}\begin{bmatrix} c\cos bt + id\cos bt + ic\sin bt - d\sin bt \\ f\cos bt + ig\cos bt + if\sin bt - g\sin bt \end{bmatrix}\end{aligned}$$

Similarly, we can write $\vec{\boldsymbol{x}}_2$ as

$$\vec{\boldsymbol{x}}_2 = e^{at}\begin{bmatrix} c\cos bt - id\cos bt - ic\sin bt - d\sin bt \\ f\cos bt - ig\cos bt - if\sin bt - g\sin bt \end{bmatrix}.$$

Because $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$ are complex conjugates, we can form two real functions $\vec{\boldsymbol{y}}_1$ and $\vec{\boldsymbol{y}}_2$ by combining them as follows

$$\begin{aligned}\vec{\boldsymbol{y}}_1 &= \frac{\vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{x}}_2}{2}\\ &= e^{at}\begin{bmatrix} c\cos bt - d\sin bt \\ f\cos bt - g\sin bt \end{bmatrix}\\ \vec{\boldsymbol{y}}_2 &= \frac{\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2}{2i}\\ &= e^{at}\begin{bmatrix} d\cos bt + c\sin bt \\ g\cos bt + f\sin bt \end{bmatrix}.\end{aligned}$$

We can now express our general solution without imaginary numbers as a linear combination of $\vec{y}_1$ and $\vec{y}_2$

$$\begin{aligned}\vec{x} &= C_1 e^{at}\begin{bmatrix} c\cos bt - d\sin bt \\ f\cos bt - g\sin bt\end{bmatrix} \\ &\quad + C_2 e^{at}\begin{bmatrix} d\cos bt + c\sin bt \\ g\cos bt + f\sin bt\end{bmatrix},\end{aligned}$$

where C_1 and C_2 are new constants. In general, for 2×2 matrices, if we have a complex eigenvalue $\lambda = a + bi$ and corresponding eigenvector $\vec{v} = \Re(\vec{v}) + i\Im(\vec{v})$, then our general solution is

$$\begin{aligned}\vec{x} &= C_1 e^{at}\left(\Re(\vec{v})\cos bt - \Im(\vec{v})\sin bt\right) \\ &\quad + C_2 e^{at}\left(\Re(\vec{v})\sin bt + \Im(\vec{v})\cos bt\right).\end{aligned}$$

If $a < 0$ the trajectories form a spiral sink (i.e. inward spiral). If $a > 0$, the trajectories form a spiral source (i.e. outward spiral). To determine which direction (clockwise or counterclockwise) the spiral is, just evaluate $\vec{x}$ at two different times t where the different times are close to each other. If $a = 0$, the trajectories are ellipses about the origin.

When solving these continuous dynamical systems with complex eigenvalues, we don't have to go through this whole process or memorize this formula for $\vec{x}$. We can follow this general process:

1. Find a complex eigenvalue λ.
2. Find the corresponding complex eigenvector $\vec{v}$.
3. Write the general solution $\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda^* t}\vec{v}^*$ with complex values.
4. Let $\vec{x}_1 = e^{\lambda t}\vec{v}$ and transform the imaginary part of the exponential into sines and cosines using Euler's formula.
5. Then the real part and imaginary parts of $\vec{x}_1$ form a real basis for $\vec{x}$ and the solution can be written as
$$\vec{x} = C_1 e^{\Re(\lambda)}\Re(\vec{x}_1) + C_2 e^{\Re(\lambda)}\Im(\vec{x}_1).$$

Example 5.3.4

Express the general solution of $\dot{\vec{x}} = A\vec{x}$ in terms of real values and describe the trajectories if

$$A = \begin{bmatrix} 4 & 2 \\ -5 & 2\end{bmatrix}.$$

We find that an eigenvalue of A is $\lambda = 3 + 3i$, and the corresponding eigenvector is $\vec{v} = \begin{bmatrix} 1+3i \\ -5\end{bmatrix}$, so our general solution with complex values is

$$\vec{x} = c_1 e^{(3+3i)t}\begin{bmatrix} 1+3i \\ -5\end{bmatrix} + c_2 e^{(3-3i)t}\begin{bmatrix} 1-3i \\ -5\end{bmatrix}.$$

Now we define $\vec{x}_1$ as the first term of $\vec{x}$ without the constant c_1, and we use Euler's formula to express it in terms of sines and cosines and then separating it into real and imaginary parts.

$$\begin{aligned}
\vec{x}_1 &= e^{(3+3i)t}\begin{bmatrix} 1+3i \\ -5 \end{bmatrix} \\
&= e^{3t}e^{i3t}\begin{bmatrix} 1+3i \\ -5 \end{bmatrix} \\
&= e^{3t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1+3i \\ -5 \end{bmatrix} \\
&= e^{3t}\begin{bmatrix} \cos 3t + i3\cos 3t + i\sin 3t - 3\sin 3t \\ -5\cos 3t - i5\sin 3t \end{bmatrix} \\
&= e^{3t}\begin{bmatrix} \cos 3t - 3\sin 3t \\ -5\cos 3t \end{bmatrix} + ie^{3t}\begin{bmatrix} 3\cos 3t + \sin 3t \\ -5\sin 3t \end{bmatrix}.
\end{aligned}$$

Now the real-valued general solution is $\vec{x} = C_1 e^{\Re(\lambda)}\Re(\vec{x}_1) + C_2 e^{\Re(\lambda)}\Im(\vec{x}_1)$ or in our case,

$$\vec{x}(t) = C_1 e^{3t}\begin{bmatrix} \cos 3t - 3\sin 3t \\ -5\cos 3t \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 3\cos 3t + \sin 3t \\ -5\sin 3t \end{bmatrix}.$$

Since $a = 3 > 0$, the trajectories form a spiral source. If we let $C_1 = C_2 = 1$ and evaluate $\vec{x}(t)$ at $t = 0$ and $t = \frac{\pi}{12}$, we find that the spiral is going outward in the clockwise direction.

5.4 Sparse Matrices

A sparse matrix is a matrix with a lot of zeros. In quantum mechanics, we often have to find the eigenvalues and eigenvectors of matrices, and often, these matrices are sparse. Knowing tricks for computing eigenvalues and eigenvectors of sparse matrices can greatly speed up your work.

Diagonal Matrices

For a diagonal matrix, the eigenvalues are the diagonal elements, and the eigenvectors are the *standard* basis vectors. For example, if your matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then your eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and your normalized eigenvectors are

$$\vec{\boldsymbol{x}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\boldsymbol{x}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{\boldsymbol{x}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Triangular Matrices

The eigenvalues of any triangular matrix (upper or lower) are just the elements on the main diagonal. For example, if your matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

then your eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 6$. The eigenvectors are not so easily found. For triangular matrices, only one eigenvector is easily found, and the rest have to be computed manually.

For an upper triangular matrix, the easy eigenvector is the the first standard basis vector, and it corresponds to the eigenvalue in the top left corner of the matrix

$$\vec{\boldsymbol{x}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is because, for example,

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For a lower triangular matrix, the eigenvector associated with the eigenvalue in the bottom right corner is the "last" standard basis vector

$$\vec{\boldsymbol{x}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Block Diagonal Matrices

For a block diagonal matrix, the eigenvalues are just the eigenvalues of the block matrices. The eigenvectors can be very easily obtained from the eigenvectors of the block matrices.

Suppose your matrix is

$$\begin{bmatrix} 13 & -1 & 0 & 0 \\ -6 & 12 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consider the block matrix in the upper left corner

$$\begin{bmatrix} 13 & -1 \\ -6 & 12 \end{bmatrix}.$$

If we calculate its eigenvalues and eigenvectors, which are fairly easy for 2×2 matrices, we find that $\lambda_1 = 15$, $\lambda_2 = 10$, and the corresponding normalized eigenvectors are

$$\vec{\boldsymbol{x}}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{\boldsymbol{x}}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For the second block matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

we obtain the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$, and the corresponding normalized eigenvectors

$$\vec{\boldsymbol{x}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \vec{\boldsymbol{x}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This tells us that the eigenvalues of our original 4×4 matrix are $\lambda_1 = 15$, $\lambda_2 = 10$, $\lambda_3 = 0$ and $\lambda_4 = 2$. Notice that for the first block matrix, there are a pair of zeros beneath each column. For the second block matrix, there are a pair of zeros above each column. To find the eigenvectors of the 4×4 matrix, we just use the ones for the 2×2 block matrices, and add the corresponding zeros to get

$$\vec{\boldsymbol{x}}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\boldsymbol{x}}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix},$$

$$\vec{x}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Note that the eigenvalue of a 1×1 matrix is just the value of the single element. The eigenvector is just the vector with a single component 1. With this knowledge, we can easily determine the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 13 & -1 & 0 \\ 0 & -6 & 12 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

We already know the eigenvalues and eigenvectors of the center block. The only difference now is that we have a single zero above and below each column, so our final eigenvectors will look a little different. For the 1×1 matrix in the top left, the eigenvalue is 7 and the eigenvector is 1. Notice that it has three zeros below it. Putting it all together, our eigenvalues are $\lambda_1 = 7$, $\lambda_2 = 15$, $\lambda_3 = 10$ and $\lambda_4 = 5$, and the corresponding eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix},$$

$$\vec{x}_3 = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

5.5 Summary: Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Any $n \times n$ matrix has an nth degree characteristic polynomial, and therefore has n eigenvalues if you account for multiplicities. If all the eigenvalues are distinct, then the eigenvectors are linearly independent and form a basis for $\mathbb{R}^n$.

A matrix is not invertible if any of its eigenvalues are 0.

For a diagonal matrix, the eigenvalues are the diagonal elements and the eigenvectors are just the standard basis vectors. For a triangular matrix, the eigenvalues are the diagonal elements, but the eigenvectors are not the standard basis vectors.

To check if a given vector $\vec{\boldsymbol{x}}$ is an eigenvector of a given matrix A, just multiply the two and see if there is a constant λ such that $A\vec{\boldsymbol{x}} = \lambda\vec{\boldsymbol{x}}$.

To check if a given value is an eigenvalue of a matrix, compute the characteristic equation of the matrix and plug in the value to see if it makes the equation true.

Matrix exponentiation can easily be done with eigenvalues and eigenvectors

$$A^p\vec{\boldsymbol{x}} = c_1\lambda_1^p\vec{\boldsymbol{v}}_1 + \cdots + c_n\lambda_n^p\vec{\boldsymbol{v}}_n,$$

The **characteristic equation** of a matrix A is found as $\det(A - \lambda I) = 0$. Find the eigenvalues of A using the characteristic equation, then find the corresponding eigenvectors by plugging the eigenvalues λ into $(A - \lambda I)\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ one by one and solving for $\vec{\boldsymbol{x}}$. For 3×3 or larger matrices, you have to put the resulting matrix into reduced echelon form to find the eigenvector. For 2×2 matrices, you can shortcut this process since you know that the rows are not linearly independent.

The **eigenspace** associated with an eigenvector is just the span of the eigenvector.

To find complex eigenvalues, we use the same process, but since we can't factor the characteristic equation, we are forced to use the quadratic formula to find its roots. The eigenvectors are found the same way, but since they come in conjugate pairs, we only have to find one of them.

Diagonalization

Two $n \times n$ matrices A and B are **similar** if and only if they can be written as $A = PBP^{-1}$. If two matrices are similar, they have the same eigenvalues, but the converse is not true.

An $n \times n$ matrix A is **diagonalizable** if and only if it has n linearly independent eigenvectors or if (but not only if) it has n distinct eigenvalues. If it is diagonalizable, it can be written in the form $A = PDP^{-1}$, where P is the matrix whose columns are the eigenvectors of A and D is a diagonal matrix whose elements are the eigenvalues of A corresponding to the eigenvectors in P.

Diagonalization can also be used to compute powers of a matrix, since $A^k = PD^kP^{-1}$, and a diagonal matrix D raised to a power is just the matrix with the diagonal elements raised to the power.

To diagonalize an $n \times n$ matrix A,

1. Find the eigenvalues of A
2. Find n linearly independent eigenvectors of A
3. Construct P and D using what we found in steps 1 and 2.

We cannot diagonalize matrices with complex eigenvalues, but we can do something very similar by finding an eigenvalue $a+bi$ and an eigenvector $\vec{\boldsymbol{v}}_1$, then writing it as

$$A = PCP^{-1},$$

where

$$P = \left[\Re(\vec{\boldsymbol{v}}_1) \quad \Im(\vec{\boldsymbol{v}}_1)\right], \qquad C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Discrete Dynamical Systems

A **discrete** dynamical system has the difference equation

$$\vec{\boldsymbol{x}}_{k+1} = A\vec{\boldsymbol{x}}_k.$$

To solve such a system,

1. Find the eigenvalues λ_1 and λ_2 of A. If they are complex, you only have to find one since the other will be its complex conjugate.
2. Find the eigenvectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ of A. If they are complex, you only have to find one since the other will be its complex conjugate.
3. Write the general solution which has the form

$$\vec{\boldsymbol{x}}_k = c_1\lambda_1^k\vec{\boldsymbol{v}}_1 + c_2\lambda_2^k\vec{\boldsymbol{v}}_2.$$

4. If given the initial conditions $\vec{\boldsymbol{x}}(0) = \vec{\boldsymbol{x}}_0$, use it to find the arbitrary constants c_1 and c_2.
5. Draw the trajectories of the general solution by first plotting the eigenvectors, then if λ are real
   - If all $|\lambda| < 1$, the origin is a sink (attractor)
   - If all $|\lambda| > 1$, the origin is a source (repellor)

- If the size of the eigenvalues are mixed, the origin is a saddle

6. If λ are complex
   - If $|\lambda| < 1$, the trajectory is an inward spiral about about the origin (attractor)
   - If $|\lambda| = 1$, the trajectory is a closed loop around the origin
   - If $|\lambda| > 1$, the trajectory is an outward spiral about the origin (repellor)
   - To determine the direction of the spiral, we simply multiply A by a convenient vector $\vec{x}_0$ and then the arrow on $\mathbb{R}^2$ drawn from $\vec{x}_0$ to $A\vec{x}_0$ tells us whether the spiral is clockwise or counterclockwise.

When plotting the trajectories for discrete dynamical systems, remember that the trajectories are dots rather than continuous lines. Be sure to indicate this on the plots.

Continuous Dynamical Systems

A **continuous** dynamical system is represented by the differential matrix equation

$$\dot{\vec{x}} = A\vec{x}.$$

To solve such a system,

1. Find the eigenvalues λ_1 and λ_2 of A. If they are complex, you only have to find one since the other will be its complex conjugate.
2. Find the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A. If they are complex, you only have to find one since the other will be its complex conjugate.
3. Write the general solution which has the form
$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$
4. If the eigenvectors are real,
   a) Plot the trajectories of the general solution, by first plotting the eigenvectors, then $\lambda > 0$ repels in the direction of $\vec{v}$, and $\lambda < 0$ attracts in the direction of $\vec{v}$.
   b) Solve for the constants c_1 and c_2 by setting $t = 0$ if given the initial conditions $\vec{x}(0) = \vec{x}_0$
5. If the eigenvalues are complex,
   a) Let $\vec{x}_1 = e^{\lambda t}\vec{v}$ and transform the imaginary part of the exponential into sines and cosines using Euler's formula.
   b) Then the real part and imaginary parts of $\vec{x}_1$ form a real basis for $\vec{x}$ and the solution can be written as
   $$\vec{x} = C_1 e^{\Re(\lambda)}\Re(\vec{x}_1) + C_2 e^{\Re(\lambda)}\Im(\vec{x}_1).$$
   c) If $\Re(\lambda) < 0$ the trajectories form a spiral sink (i.e. inward spiral). If $\Re(\lambda) > 0$, the trajectories form a spiral source (i.e. outward spiral). To determine which direction (clockwise or counterclockwise) the spiral is, just evaluate $\vec{x}$ at two different times t where the different times are close to each other. If $\Re(\lambda) = 0$, the trajectories are ellipses about the origin.

For 2×2 matrices, we can use the table below.

	Discrete Dynamical Systems	Continuous Dynamical Systems
Equation	$\vec{x}_k = A\vec{x}_{k-1}$	$\dot{\vec{x}} = A\vec{x}$
Origin is Sink	$\|\lambda\| < 1$ for all λ	$\Re(\lambda) < 0$ for all λ
Origin is Source	$\|\lambda\| > 1$ for all λ	$\Re(\lambda) > 0$ for all λ
Origin is Saddle	$\|\lambda\| > 1$ other is $\|\lambda\| < 1$	$\Re(\lambda) > 0$ other is $\Re(\lambda) < 0$
Spiral	Complex λ	Complex λ

Chapter 6

Orthogonality

The **inner product** or **dot product** of two vectors $\vec{u}$ and $\vec{v}$ is the matrix product

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u}.$$

Two vectors are **orthogonal** if and only if their dot product is zero.

Example 6.0.1

Calculate the dot product of $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix}.$$

Calculating the dot product, we have that

$$\begin{aligned} \vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} &= \begin{bmatrix} 3 & -5 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix} \\ &= (3)(-4) + (-5)(10) + (6)(2) \\ &= -50. \end{aligned}$$

The **norm** or length of a vector is the square root of the dot product with itself. The norm of $\vec{v}$, denoted $|\vec{v}|$

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the dot product can also be defined as

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta,$$

where θ is the angle between the two vectors. The angle between $\vec{u}$ and $\vec{v}$ can therefore be calculated as

$$\theta = \cos^{-1} \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \right).$$

Example 6.0.2

Calculate the norm of

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}.$$

The norm is

$$|\vec{\boldsymbol{u}}| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{(3)^2 + (-5)^2 + (6)^2} = \sqrt{70}.$$

A **unit vector** is a vector with norm equal to one. A vector divided by its norm is the unit vector in that direction. If $\vec{\boldsymbol{v}}$ is some vector, and $\hat{v}$ is the unit vector in the same direction, then

$$\hat{v} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}.$$

Example 6.0.3

Find the unit vector in the direction $\vec{\boldsymbol{u}}$ if

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}.$$

Earlier, we calculated $|\vec{\boldsymbol{u}}| = \sqrt{70}$, so the unit vector in the direction of $\vec{\boldsymbol{u}}$ is

$$\hat{u} = \frac{\vec{\boldsymbol{u}}}{|\vec{\boldsymbol{u}}|} = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{70}} \\ -\frac{5}{\sqrt{70}} \\ \frac{6}{\sqrt{70}} \end{bmatrix}.$$

To check, calculate the norm of $\hat{u}$ to ensure that it is 1.

The **distance** between two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ is defined as the norm of the vector joining their endpoints. That is,

$$D(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = |\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}| = |\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}|.$$

Two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are orthogonal if and only if

$$|\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}|^2 = |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2.$$

This is just the Pythagorean theorem and can easily be shown by letting $|\vec{\boldsymbol{u}}|$ and $|\vec{\boldsymbol{v}}|$ form the bases of a right triangle. Then the hypotenuse is given by $|\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}|$.

Example 6.0.4

Calculate the distance between vectors $\vec{u}$ and $\vec{v}$ if

$$\vec{u} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix}.$$

To calculate the distance, we first calculate the difference

$$\vec{u} - \vec{v} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix} - \begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -15 \\ 4 \end{bmatrix},$$

then we take the norm

$$|\vec{u} - \vec{v}| = \sqrt{(7)^2 + (-15)^2 + (4)^2} = \sqrt{290}.$$

If a vector $\vec{v}$ is perpendicular to every vector in a subspace W, then $\vec{v}$ is orthogonal to W, and is said to be in the **orthogonal complement** of W, denoted $W^{\perp}$. For example, consider a plane W in $\mathbb{R}^3$, then $W^{\perp}$ is the set of all vectors that are orthogonal to the plane W.

6.1 Orthogonal Projections

A set of vectors is an **orthogonal set** if the dot product of any pair is zero with the exception of those paired with themself. In other words, the set $\{\vec{u}_1, \dots, \vec{u}_n\}$ is orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$.

Any set of orthogonal nonzero vectors is linearly independent.

An orthogonal set $\{\vec{u}_1, \dots, \vec{u}_n\}$, since it is linearly independent, forms an orthogonal basis for span$\{\vec{u}_1, \dots, \vec{u}_n\}$. An **orthogonal basis** is simply an orthogonal set of basis vectors. A set of basis vectors does not have to be orthogonal, but an orthogonal basis is typically easier to work with than a non-orthogonal basis.

If $U = \{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthogonal basis for some subspace, and if $\vec{x}$ is in that subspace, then we know that $\vec{x}$ can be represented as a linear combination of the vectors in U. That is,

$$\vec{x} = c_1\vec{u}_1 + \cdots + c_i\vec{u}_i + \cdots + c_n\vec{u}_n.$$

If U was a typical basis, then to find the coefficients c_i, we would have to put the above equation into an augmented matrix

$$[\vec{u}_1 \cdots \vec{u}_n \,|\, \vec{x}],$$

and solve for the coefficients. However, since U is an orthogonal basis, we can find the coefficients more easily. To find c_i, we just "dot" the equation with $\vec{u}_i$

$$\begin{aligned} \vec{u}_i \cdot \vec{x} &= \vec{u}_i \cdot (c_1\vec{u}_1 + \cdots + c_i\vec{u}_i + \cdots + c_n\vec{u}_n) \\ &= c_1\vec{u}_i \cdot \vec{u}_1 + \cdots + c_i\vec{u}_i \cdot \vec{u}_i + \cdots + c_n\vec{u}_i \cdot \vec{u}_n. \end{aligned}$$

Since U is an orthogonal basis, then $\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_j = 0$ except when $i = j$, so all the terms on the right are zero except for the one in which the basis vector is dotted with itself. So we get

$$\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{x}} = c_i \vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_i,$$

which allows us to calculate each coefficient as

$$c_i = \frac{\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{x}}}{\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_i}.$$

Computationally, this is much easier than using an augmented matrix and row reducing it.

Example 6.1.1

Express

$$\vec{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

as a linear combination of the orthogonal basis vectors

$$\vec{\boldsymbol{u}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{\boldsymbol{u}}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{\boldsymbol{u}}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We know that we can write

$$\vec{\boldsymbol{x}} = c_1\vec{\boldsymbol{u}}_1 + c_2\vec{\boldsymbol{u}}_2 + c_3\vec{\boldsymbol{u}}_3,$$

and to find the coefficients we compute

$$\begin{aligned} c_1 &= \frac{\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{x}}}{\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_1} = \frac{6}{3} = 2 \\ c_2 &= \frac{\vec{\boldsymbol{u}}_2 \cdot \vec{\boldsymbol{x}}}{\vec{\boldsymbol{u}}_2 \cdot \vec{\boldsymbol{u}}_2} = \frac{3}{6} = \frac{1}{2} \\ c_3 &= \frac{\vec{\boldsymbol{u}}_3 \cdot \vec{\boldsymbol{x}}}{\vec{\boldsymbol{u}}_3 \cdot \vec{\boldsymbol{u}}_3} = \frac{-1}{2} = -\frac{1}{2}, \end{aligned}$$

so our solution is

$$\vec{\boldsymbol{x}} = 2\vec{\boldsymbol{u}}_1 + \frac{1}{2}\vec{\boldsymbol{u}}_2 - \frac{1}{2}\vec{\boldsymbol{u}}_3.$$

Recall from elementary calculus, that the **orthogonal projection** of $\vec{\boldsymbol{u}}$ onto a vector $\vec{\boldsymbol{v}}$ is given by

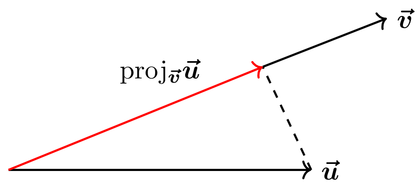
$$\operatorname{proj}_{\vec{\boldsymbol{v}}} \vec{\boldsymbol{u}} = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}}\vec{\boldsymbol{v}},$$

and illustrated by the diagram below.

Notice that the projection can also be written as

$$\operatorname{proj}_{\vec{\boldsymbol{v}}} \vec{\boldsymbol{u}} = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|^2}\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}\hat{v},$$

where $\hat{v}$ is the unit vector in the direction of $\vec{\boldsymbol{v}}$.



Notice that $\text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}}$ is parallel to $\vec{\boldsymbol{v}}$ and can therefore be written as

$$\text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}} = \alpha\vec{\boldsymbol{v}},$$

where α is some scaling constant. Secondly, the vector $\vec{\boldsymbol{u}} - \text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}}$ (i.e. the dotted line), is a vector that is orthogonal to $\vec{\boldsymbol{v}}$. If we call this orthogonal vector $\vec{\boldsymbol{z}}$, then

$$\vec{\boldsymbol{z}} = \vec{\boldsymbol{u}} - \text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}}.$$

So if we're asked to write a given vector $\vec{\boldsymbol{u}}$ in terms of a vector in the direction of $\vec{\boldsymbol{v}}$ and a vector orthogonal to $\vec{\boldsymbol{v}}$, then we write

$$\begin{aligned}\vec{\boldsymbol{u}} &= \alpha\vec{\boldsymbol{v}} + \vec{\boldsymbol{z}}\\ &= \text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}} + (\vec{\boldsymbol{u}} - \text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}})\\ &= \frac{\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}}}{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}}\vec{\boldsymbol{v}} + \left(\vec{\boldsymbol{u}} - \frac{\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}}}{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}}\vec{\boldsymbol{v}}\right).\end{aligned}$$

Example 6.1.2

Given

$$\vec{\boldsymbol{v}} = \begin{bmatrix}1\\2\end{bmatrix}, \qquad \vec{\boldsymbol{u}} = \begin{bmatrix}3\\4\end{bmatrix},$$

find the projection of $\vec{\boldsymbol{u}}$ onto $\vec{\boldsymbol{v}}$ and express $\vec{\boldsymbol{u}}$ as the sum of a vector parallel to $\vec{\boldsymbol{v}}$ and a vector orthogonal to $\vec{\boldsymbol{v}}$.

The projection of $\vec{\boldsymbol{u}}$ onto $\vec{\boldsymbol{v}}$ is

$$\text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}} = \frac{\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}}}{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}}\vec{\boldsymbol{v}} = \frac{11}{5}\vec{\boldsymbol{v}}.$$

Notice that this is a scaled version of $\vec{\boldsymbol{v}}$, so it is parallel to $\vec{\boldsymbol{v}}$. The vector

$$\vec{\boldsymbol{z}} = \vec{\boldsymbol{u}} - \text{proj}_{\vec{\boldsymbol{v}}}\,\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}} - \frac{11}{5}\vec{\boldsymbol{v}},$$

is orthogonal to $\vec{v}$, and we can write

$$\begin{aligned}
\vec{u} &= \operatorname{proj}_{\vec{v}} \vec{u} + (\vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}) \\
&= \frac{11}{5}\vec{v} + \left(\vec{u} - \frac{11}{5}\vec{v}\right) \\
&= \frac{11}{5}\begin{bmatrix} 1 \\ \\ 2 \end{bmatrix} + \left(\begin{bmatrix} 3 \\ \\ 4 \end{bmatrix} - \frac{11}{5}\begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}\vec{v}\right) \\
&= \begin{bmatrix} \frac{11}{5} \\ \\ \frac{22}{5} \end{bmatrix} + \begin{bmatrix} \frac{4}{5} \\ \\ -\frac{2}{5} \end{bmatrix},
\end{aligned}$$

where the first vector is parallel to $\vec{v}$ and the second is orthogonal to $\vec{v}$.

A set of vectors is an **orthonormal** set if it is an *orthogonal set* of *unit vectors*. Orthonormal simply means the vectors are all of length one in addition to being orthogonal. To show that a set of vectors is orthonormal, show that they're orthogonal, that is, their pairwise dot products are zero, and that they are all of unit length.

An **orthogonal matrix** is a matrix with orthonormal columns. A better term would be "orthonormal matrix", but they are commonly called "orthogonal matrices".

To show that a matrix is orthogonal, just show that the columns are a set of orthonormal vectors. For example, the identity matrices are all orthogonal. You could also multiply U^TU and show that it equals I.

Theorem: An $m \times n$ matrix U is orthogonal if and only if $U^TU = I$. In order for a matrix U to be orthogonal, it must be square or have more rows than columns. In order for the columns to be orthonormal, they must be linearly independent, so U cannot have more columns than rows.

If U is an orthogonal $m \times n$ matrix, and $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then U has the following properties:

- $|U\vec{u}| = |\vec{u}|$
- $(U\vec{u}) \cdot (U\vec{v}) = \vec{u} \cdot \vec{v}$
- $(U\vec{u}) \cdot (U\vec{v}) = 0, \text{ IFF } \vec{u} \cdot \vec{v} = 0$

These three are essentially equivalent, and the second one can be proved as follows

$$\begin{aligned}
(U\vec{u}) \cdot (U\vec{v}) &= (U\vec{u})^T(U\vec{v}) \\
&= \vec{u}^T U^T (U\vec{v}) \\
&= \vec{u}^T I \vec{v} \\
&= \vec{u}^T \vec{v} \\
&= \vec{u} \cdot \vec{v}.
\end{aligned}$$

If U is a *square* invertible and orthogonal $n \times n$ matrix, then $U^{-1} = U^T$. In such a matrix, both the columns and the rows are orthonormal.

Up to this point, when calculating orthogonal projections, we've only been considering the projection of a vector onto the span of another vector. That is, we've been calculating the projection of a vector onto a subspace of dimension one. Now, we will generalize the idea of projections to the projection of a vector onto a subspace of any dimension.

If $\vec{\boldsymbol{u}}$ is a vector in $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$, then the projecton of $\vec{\boldsymbol{u}}$ onto W is denoted $\operatorname{proj}_W \vec{\boldsymbol{u}}$. For example, if $\vec{\boldsymbol{u}}$ is in $\mathbb{R}^3$ and W is a plane in $\mathbb{R}^3$, then $\operatorname{proj}_W \vec{\boldsymbol{u}}$ is the projection of $\vec{\boldsymbol{u}}$ onto the plane W. Keep in mind that $\operatorname{proj}_W \vec{\boldsymbol{u}}$ lies in W.

Theorem: If $\{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_p\}$ is an *orthogonal* basis of the subspace W which lies in $\mathbb{R}^n$, then

$$\begin{aligned}\operatorname{proj}_W \vec{\boldsymbol{y}} &= \operatorname{proj}_{\vec{\boldsymbol{u}}_1} \vec{\boldsymbol{y}} + \cdots + \operatorname{proj}_{\vec{\boldsymbol{u}}_p} \vec{\boldsymbol{y}} \\ &= \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_1}{\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_1}\vec{\boldsymbol{u}}_1 + \cdots + \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_p}{\vec{\boldsymbol{u}}_p \cdot \vec{\boldsymbol{u}}_p}\vec{\boldsymbol{u}}_p.\end{aligned}$$

In other words, the projection of a vector onto a subspace is just the sum of the projections of the vector onto the orthogonal basis vectors of the subspace. Note: If $\vec{\boldsymbol{y}}$ is in W, then $\operatorname{proj}_W \vec{\boldsymbol{y}}$ is just $\vec{\boldsymbol{y}}$.

If $\{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_p\}$ is an *orthonormal* basis of the subspace W which lies in $\mathbb{R}^n$, then

$$\operatorname{proj}_W \vec{\boldsymbol{y}} = (\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 + \cdots + (\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_p)\vec{\boldsymbol{u}}_p.$$

Notice that this expression is a little simpler than the above one because we're now dealing with orthogonal basis vectors that are also unit vectors, so $\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_i = 1$.

Example 6.1.3

Find the projection of

$$\vec{\boldsymbol{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

in $\mathbb{R}^3$ onto the xy-plane.

In this case, the xy-plane is our subspace W. We know that the standard basis vectors $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$ form an orthogonal basis for the xy-plane. So

$$\begin{aligned}\operatorname{proj}_W \vec{\boldsymbol{y}} &= \operatorname{proj}_{\vec{\boldsymbol{e}}_1} \vec{\boldsymbol{y}} + \operatorname{proj}_{\vec{\boldsymbol{e}}_2} \vec{\boldsymbol{y}} \\ &= \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{e}}_1}{\vec{\boldsymbol{e}}_1 \cdot \vec{\boldsymbol{e}}_1}\vec{\boldsymbol{e}}_1 + \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{e}}_2}{\vec{\boldsymbol{e}}_2 \cdot \vec{\boldsymbol{e}}_2}\vec{\boldsymbol{e}}_2 \\ &= \vec{\boldsymbol{e}}_1 + 2\vec{\boldsymbol{e}}_2 \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.\end{aligned}$$

Example 6.1.4

Find the projection of $\vec{\boldsymbol{y}}$ onto $W = \operatorname{span}\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2\}$, if

$$\vec{\boldsymbol{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{\boldsymbol{u}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{\boldsymbol{u}}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

then write $\vec{\boldsymbol{y}}$ as a vector in W plus a vector orthogonal to W.

Taking the dot product $\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_2 = 0$, we verify that $\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2\}$ is an orthogonal basis, so

$$\begin{aligned}
\operatorname{proj}_W \vec{\boldsymbol{y}} &= \operatorname{proj}_{\vec{\boldsymbol{u}}_1} \vec{\boldsymbol{y}} + \operatorname{proj}_{\vec{\boldsymbol{u}}_2} \vec{\boldsymbol{y}} \\
&= \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_1}{\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_1}\vec{\boldsymbol{u}}_1 + \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_2}{\vec{\boldsymbol{u}}_2 \cdot \vec{\boldsymbol{u}}_2}\vec{\boldsymbol{u}}_2 \\
&= 2\vec{\boldsymbol{u}}_1 + \frac{1}{2}\vec{\boldsymbol{u}}_2 \\
&= \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}.
\end{aligned}$$

This is a vector in W. Then $\vec{\boldsymbol{y}} - \operatorname{proj}_W \vec{\boldsymbol{y}}$ is a vector orthogonal to W. So

$$\begin{aligned}
\vec{\boldsymbol{y}} &= \operatorname{proj}_W \vec{\boldsymbol{y}} + (\vec{\boldsymbol{y}} - \operatorname{proj}_W \vec{\boldsymbol{y}}) \\
&= \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

The *distance* between a vector and a subspace is defined as the distance between the vector and its projection onto the subspace. So given a vector $\vec{\boldsymbol{y}}$ and a subspace W,

$$D(\vec{\boldsymbol{y}}, W) = D(\vec{\boldsymbol{y}}, \operatorname{proj}_W \vec{\boldsymbol{y}}) = |\vec{\boldsymbol{y}} - \operatorname{proj}_W \vec{\boldsymbol{y}}|.$$

This makes sense, because the shortest distance between the end of a vector and a plane (for example), is the line from the endpoint of the vector to a point in the plane such that the line is perpendicular to the plane.

6.2 The Gram-Schmidt Process

The Gram-Schmidt process allows us to compute orthogonal basis vectors given a set of non-orthogonal basis vectors. It is an algorithmic approach using projections like we used before. For example, given non-orthogonal basis vectors $\{\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{x}}_3\}$ for $\mathbb{R}^3$, we we want to find an orthogonal set of basis vectors $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3\}$. We start by defining

$$\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{x}}_1.$$

To find $\vec{\boldsymbol{v}}_2$, we need a vector that is in $\operatorname{span}\{\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{x}}_3\}$ but also orthogonal to the previously found vector $\vec{\boldsymbol{v}}_1$. To do that, recall that $\vec{\boldsymbol{x}}_2 - \operatorname{proj}_{\vec{\boldsymbol{v}}_1} \vec{\boldsymbol{x}}_2$ is a vector that is orthogonal to $\vec{v}_1$, so we let

$$\vec{v}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1.$$

Now, the vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and span a plane. Our third vector, must also be in $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$, and it must be orthogonal to $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. Recall the process of finding a vector orthogonal to a subspace given orthogonal basis vectors for the subspace. In this case, we know that $\vec{v}_1$ and $\vec{v}_2$ form an orthogonal basis for $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. If we say $W = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, then we know that $\vec{x}_3 - \operatorname{proj}_W \vec{x}_3$, is orthogonal to W, and we know that $\operatorname{proj}_W \vec{x}_3 = \operatorname{proj}_{\vec{v}_1} \vec{x}_3 + \operatorname{proj}_{\vec{v}_2} \vec{x}_3$, so

$$\begin{aligned} \vec{v}_3 &= \vec{x}_3 - \operatorname{proj}_{\vec{v}_1} \vec{x}_3 - \operatorname{proj}_{\vec{v}_2} \vec{x}_3 \\ &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2. \end{aligned}$$

We now have three orthogonal vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that span the same space as $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. To find an *orthonormal* basis, we just have to normalize each of these vectors.

This same process extends to bases of any dimension. In general, given a non-orthogonal set of basis vectors $\{\vec{x}_1, \dots, \vec{x}_n\}$, an orthogonal set of basis vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ spanning the same space can be found as

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\ \vdots &= \vdots \\ \vec{v}_n &= \vec{x}_n - \sum_{i=1}^{n-1} \frac{\vec{x}_n \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}\vec{v}_i. \end{aligned}$$

Example 6.2.1

Using the Gram-Schmidt process, find an orthonormal set of basis vectors given the basis vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Using the formulas

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1, \end{aligned}$$

we get

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{1}{2}\vec{v}_1 \end{aligned}$$

so

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}.$$

Normalizing them gives us

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Taking their inner product (i.e. dot product) yields zero, confirming that they are indeed orthogonal.

Advanced

The **Gram-Schmidt orthogonalization theorem** states that: Let V be a vector space with positive definite inner product. Let $v_1, v_2, \ldots, v_k$ be a linearly independent set in V. Then there exists an orthogonal set $u_1, u_2, \ldots, u_k$ such that $(v_1, u_i) > 0$ and $\text{span}\{v_1, v_2, \ldots, v_i\} = \text{span}\{u_1, u_2, \ldots, u_i\}$ for all i from 1 to k. In other words, given a linearly independent set of basis vectors, we can find an orthogonal set of basis vectors.

We can also apply orthogonalization to polynomials on $[0, 1]$. The polynomials $v_0 = 1$, $v_1 = x$, and $v_2 = x^2$ form a basis for the space of polynomials of integer degree ≤ 2. The polynomials v_0, v_1, and v_2 are not an orthogonal basis, but we can find an orthogonal basis using the Gram-Schmidt process. Recall that the inner product for functions is defined as

$$(f, g) = \int_a^b f^*(x) g(x)\, dx.$$

Using the Gram-Schmidt process, we have that

$$u_0 = v_0 = 1,$$

and

$$u_1 = v_1 - \frac{(v_1, u_0)}{||u_0||^2} u_0.$$

Doing the integrals, we get

$$\begin{aligned} (v_1, u_0) &= \int_0^1 x\, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2} \\ ||u_0||^2 &= \int_0^1 1\, dx = x \Big|_0^1 = 1, \end{aligned}$$

so

$$u_1 = x - \frac{1}{2}.$$

Next, the formula for u_2 is

$$u_2 = v_2 - \frac{(v_2, u_0)}{||u_0||^2} u_0 - \frac{(v_2, u_1)}{||u_1||^2} u_1.$$

Doing the integrals gives us

$$\begin{aligned}(v_2, u_0) &= \int_0^1 x^2\,dx = \frac{1}{3}\\ (v_2, u_1) &= \int_0^1 x^2\left(x-\frac{1}{2}\right)dx = \frac{1}{12}\\ ||u_1||^2 &= \int_0^1 \left(x-\frac{1}{2}\right)^2 dx = \frac{1}{12},\end{aligned}$$

so

$$u_2 = x^2 - x + \frac{1}{6}.$$

Orthogonalization of polynomials on $[-1, 1]$ leads to special orthogonal polynomials called **Legendre polynomials**.

6.3 Least Squares Solutions and Curve Fitting

Given an inconsistent (i.e. unsolvable) equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, we are often interested in obtaining the vector $\vec{\boldsymbol{x}}$ that is closest to $\vec{\boldsymbol{b}}$. In other words, we want to minimize the distance between $A\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ by finding some special $\vec{\boldsymbol{x}}$ such that $D(\vec{\boldsymbol{b}}, A\vec{\boldsymbol{x}}) = ||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||$ is as small as possible. If $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ had an exact solution, this distance would be zero. The $\vec{\boldsymbol{x}}$ that minimizes this distance is called the **least squares solution**. If we denote the vector that minimizes the distance $\vec{\boldsymbol{x}}_m$, then

$$||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}_m|| \leq ||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||,$$

for all possible vectors $\vec{\boldsymbol{x}}$.

If A is an $m \times n$ matrix, then $\vec{\boldsymbol{x}}$ is in $\mathbb{R}^n$ and $\vec{\boldsymbol{b}}$ is in $\mathbb{R}^m$. Notice that the vector resulting from the product $A\vec{\boldsymbol{x}}$ is in the subspace Col A. The distance between a vector and a subspace is given by the orthogonal projection of the vector onto that subspace, so we know that the distance $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}_m||$ is $||\vec{\boldsymbol{b}} - \text{proj}_{Col\,A}\vec{\boldsymbol{b}}||$. Now, recall that a vector minus the orthogonal projection of that vector onto a subspace gives a vector that is orthogonal to the subspace, so $\vec{\boldsymbol{b}} - \text{proj}_{Col\,A}\vec{\boldsymbol{b}}$ is orthogonal to Col A. In other words, the dot product of each column of A, with the vector $\vec{\boldsymbol{b}} - \text{proj}_{Col\,A}\vec{\boldsymbol{b}}$ yields zero

$$\begin{aligned}A^T(\vec{\boldsymbol{b}} - \text{proj}_{Col\,A}\vec{\boldsymbol{b}}) &= 0\\ A^T\vec{\boldsymbol{b}} - A^T\text{proj}_{Col\,A}\vec{\boldsymbol{b}} &= 0\\ A^T\vec{\boldsymbol{b}} &= A^T\text{proj}_{Col\,A}\vec{\boldsymbol{b}}\\ A^T\vec{\boldsymbol{b}} &= A^T(A\vec{\boldsymbol{x}}_m).\end{aligned}$$

Now, we have a new equation

$$A^T A\vec{\boldsymbol{x}}_m = A^T\vec{\boldsymbol{b}},$$

which we know has a solution and that solution $\vec{\boldsymbol{x}}_m$ is the least squares solution to the equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.

Example 6.3.1

Solve $A\vec{x} = \vec{b}$. If there is no solution, find the least squares solution.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

Notice first of all that $\vec{b}$ is in $\mathbb{R}^3$ and the equation $A\vec{x} = \vec{b}$ will not have a solution unless $\vec{b}$ happens to lie in the plane formed by the columns of A. If we put the system into an augmented matrix and row reduce, we find that the system is inconsistent, that is, there is no $\vec{x}$ that makes the equation true. We will therefore, find the next best thing–the least squares solution.

Calculating A^T and performing the multiplication A^TA, we get

$$A^TA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}.$$

Next, we compute $A^T\vec{b}$ to get

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Finally, to solve $(A^TA)\vec{x}_m = (A^T\vec{b})$ we create the augmented matrix

$$\left[\begin{array}{cc|c} 14 & 32 & 4 \\ 32 & 77 & 4 \end{array}\right],$$

then reduce it to the RREF

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{10}{3} \\ 0 & 1 & -\frac{4}{3} \end{array}\right].$$

So our least squares solution is

$$\vec{x}_m = \begin{bmatrix} \frac{10}{3} \\ -\frac{4}{3} \end{bmatrix}.$$

This is the vector $\vec{x}$ that makes $A\vec{x}$ as close to $\vec{b}$ as possible.

In this case, the columns of A are independent, so $A\vec{x} = \text{Col } A$ is a plane in $\mathbb{R}^3$, and the vector $\vec{x}_m$ is the vector such that $A\vec{x}_m$ is as close to $\vec{b}$ as possible.

If the reduced echelon form of $\left[\, A^T A \,|\, A^T\vec{\boldsymbol{b}} \,\right]$ has free variables, then the least squares solution $\vec{\boldsymbol{x}}_m$ will not be unique. That is okay. There may be cases where you have an infinite number of $\vec{\boldsymbol{x}}_m$ that minimize $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||$. For example, if A is 3×2, as in the example above, and the columns are not linearly independent, then the columns of A form a subspace that is only a line rather than a plane. Then if $\vec{\boldsymbol{b}}$ is not parallel to this line, the equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ will have no solutions. The least squares solution will be a unique vector $A\vec{\boldsymbol{x}}_m$, however, there may be infinite $\vec{\boldsymbol{x}}_m$ that give this unique vector $A\vec{\boldsymbol{x}}_m$.

Instead of solving $A^T A\vec{\boldsymbol{x}}_m = A^T\vec{\boldsymbol{b}}$ to find the least squares solution, we could also find the projection $\text{proj}_{Col\ A}\vec{\boldsymbol{b}}$ and factor out A since $\text{proj}_{Col\ A}\vec{\boldsymbol{b}} = A\vec{\boldsymbol{x}}_m$. However, this only works if the columns of A are orthogonal. If they are not, we would have to find an orthogonal basis for Col A using the Gram-Schmidt process before finding the orthogonal projection.

Example 6.3.2

Find the least squares solution of $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ given

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \qquad \vec{\boldsymbol{b}} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

By taking the dot products, we note that the columns of A are not orthogonal, but we can use the Gram-Schmidt process to compute an orthogonal basis for Col A. For the first vector, $\vec{\boldsymbol{v}}_1$, we just use the first column of A, denoted $\vec{\boldsymbol{a}}_1$. For the second vector, we use

$$\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{a}}_2 - \frac{\vec{\boldsymbol{a}}_2 \cdot \vec{\boldsymbol{v}}_1}{\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{v}}_1}\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{a}}_2 - \frac{16}{7}\vec{\boldsymbol{v}}_1 = \begin{bmatrix} \frac{12}{7} \\ \frac{3}{7} \\ -\frac{6}{7} \end{bmatrix}.$$

So $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ form an orthogonal basis for A. Now we can compute the relevant projection as

$$\begin{aligned} \text{proj}_{Col\ A}\vec{\boldsymbol{b}} &= \text{proj}_{\vec{\boldsymbol{v}}_1}\vec{\boldsymbol{b}} + \text{proj}_{\vec{\boldsymbol{v}}_2}\vec{\boldsymbol{b}} \\ &= \frac{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_1}{\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{v}}_1}\vec{\boldsymbol{v}}_1 + \frac{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_2}{\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{v}}_2}\vec{\boldsymbol{v}}_2 \\ &= \frac{2}{7}\vec{\boldsymbol{v}}_1 - \frac{4}{3}\vec{\boldsymbol{v}}_2. \end{aligned}$$

Writing this in terms of the columns of A, we get

$$\begin{aligned} &= \frac{2}{7}\vec{\boldsymbol{a}}_1 - \frac{4}{3}\left(\vec{\boldsymbol{a}}_2 - \frac{16}{7}\vec{\boldsymbol{a}}_1\right) \\ &= \frac{10}{3}\vec{\boldsymbol{a}}_1 - \frac{4}{3}\vec{\boldsymbol{a}}_2 \\ &= [\vec{\boldsymbol{a}}_1\ \vec{\boldsymbol{a}}_2]\begin{bmatrix} \frac{10}{3} \\ -\frac{4}{3} \end{bmatrix} \\ &= A\vec{\boldsymbol{x}}_m. \end{aligned}$$

So we get the same least squares solution

$$\vec{\boldsymbol{x}}_m = \begin{bmatrix} \frac{10}{3} \\ -\frac{4}{3} \end{bmatrix},$$

as we did using the other method.

The **least squares error** is the actual value of $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}_m||$ where $\vec{\boldsymbol{x}}_m$ is the vector that minimizes $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||$.

To find the least squares error of $A\vec{\boldsymbol{x}} = b$,

1. Compute the vector $\vec{\boldsymbol{x}}$ that minimizes $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||$
2. Then compute $||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}||$.

Example 6.3.3

Compute the least squares error of the least squares solution found in the previous two examples.

The least squares error is

$$\begin{aligned} ||\vec{\boldsymbol{b}} - A\vec{\boldsymbol{x}}_m|| &= \left|\left| \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \frac{10}{3} \\ -\frac{4}{3} \end{bmatrix} \right|\right| \\ &= \left|\left| \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \right|\right| \\ &= \left|\left| \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right|\right| \\ &= \sqrt{(-1)^2 + (2)^2 + (-1)^2} = \sqrt{6}. \end{aligned}$$

Given a set of points $(x_1, y_1), \ldots, (x_n, y_n)$ we are often interested in finding a linear best fit line to model the data. One of the easiest ways to calculate a linear best fit line is to use a **least squares line**.

A line in $\mathbb{R}^2$ has the form $y = mx + b$. For our purposes, we'll use the notation

$$y = b_0 + b_1 x.$$

If we have a perfect fit line, one that goes through every one of our points, then we have the system of equations

$$\begin{aligned} y_1 &= b_0 + b_1 x_1 \\ \vdots &= \vdots \\ y_n &= b_0 + b_1 x_n. \end{aligned}$$

Notice that b_0 and b_1 are constants, the only values changing for the different points are y and x. We can write this system of equations as the matrix equation

$$X\vec{\boldsymbol{b}} = \vec{\boldsymbol{y}},$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \qquad \vec{\boldsymbol{b}} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \qquad \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

For a perfect fit line, the solution $\vec{\boldsymbol{b}}$ to $X\vec{\boldsymbol{b}} = \vec{\boldsymbol{y}}$ allows us to write the equation of the line as $y = b_0 + b_1 x$. In reality, however, the line is not a perfect fit, that is, the points don't lie exactly on a straight line, so there is no solution $\vec{\boldsymbol{b}}$ to the matrix equation $X\vec{\boldsymbol{b}} = \vec{\boldsymbol{y}}$. However, we can find the best fit line by finding the least squares solution to the equation. In other words, we want to find the $\vec{\boldsymbol{b}}$ that minimizes $||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$.

Assuming that we only have errors in y, we want to find a line that minimizes the vertical distances between the points and the best fit line. The vertical distance between a point and the best fit line is called the **residual** for that point, and we want to find the line that minimizes the residuals. If we calculate the residual for each point, we get the vector

$$\vec{\boldsymbol{r}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix},$$

where r_i is the residual of point i. Then the magnitude of the residual vector is

$$||\vec{\boldsymbol{r}}|| = \sqrt{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}} = \sqrt{\sum_i^n r_i^2}.$$

It turns out that $||\vec{\boldsymbol{r}}|| = ||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$.

Like before, we minimize $||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$ by solving

$$X^T X\vec{\boldsymbol{b}}_m = X^T\vec{\boldsymbol{y}}.$$

You can also use least squares to fit data to a different kind of curve. For example, if your data points seem to be scattered along a quadratic curve with equation

$$y = b_0 + b_1 x + b_2 x^2,$$

then your system of equations for a perfect fit will be

$$\begin{aligned} y_1 &= b_0 + b_1 x_1 + b_2 x_1^2 \\ \vdots &= \vdots \\ y_n &= b_0 + b_1 x_n + b_2 x_n^2. \end{aligned}$$

We can again express this is $\vec{\boldsymbol{y}} = X\vec{\boldsymbol{b}}$, but now,

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}, \quad \vec{\boldsymbol{b}} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}, \quad \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Then for the least squares fit line, the same process of solving $(X^T X)\vec{\boldsymbol{b}}_m = (X^T\vec{\boldsymbol{y}})$ is used to minimize $||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$.

Keep in mind that this method of calculating the least squares line is equivalent to minimizing the sum of the squares of the residuals and the residuals are measured in the y-direction. That is, this method is valid when the x-values are exact and the measurement error is only in the y-component. If the measurement errors are in x instead of y, then the coordinates just have to be switched prior to computing the least squares line. If there are errors in both x and y, then the residuals should be measured orthogonally to the best fit line, and a different method of computing the line must be used.

Example 6.3.4

Find the least squares line for the points $(1, 2), (3, 3), (4, 5)$ and $(6, 6)$.

We want to find a $\vec{\boldsymbol{b}}$ that minimizes $||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$, and we do that by solving $(X^T X)\vec{\boldsymbol{b}}_m = (X^T\vec{\boldsymbol{y}})$. In our case,

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}, \vec{\boldsymbol{b}} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad \vec{\boldsymbol{y}} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix}.$$

Doing the matrix multiplications, we get

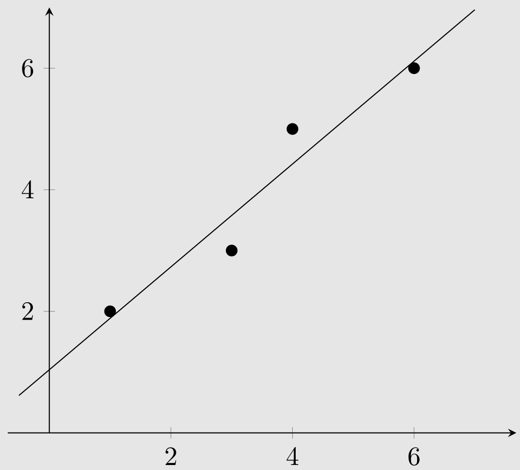
$$\begin{aligned} (X^T X)\vec{\boldsymbol{b}}_m &= (X^T\vec{\boldsymbol{y}}) \\ \begin{bmatrix} 4 & 14 \\ 14 & 62 \end{bmatrix} \vec{\boldsymbol{b}}_m &= \begin{bmatrix} 16 \\ 67 \end{bmatrix}. \end{aligned}$$

Putting this into an augmented matrix and solving it, we find that

$$\vec{\boldsymbol{b}}_m = \begin{bmatrix} \frac{27}{26} \\ \frac{11}{13} \end{bmatrix},$$

so the equation of the least squares line is

$$y = \frac{27}{26} + \frac{11}{13}x.$$


6
4
2
2
4
6

6.4 Summary: Orthogonality

The **inner product** or **dot product** of two vectors $\vec{u}$ and $\vec{v}$ is the matrix product

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u}.$$

Two vectors are **orthogonal** if and only if their dot product is zero. The **norm** or length of a vector is the square root of the dot product with itself.

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}.$$

The **distance** between two vectors $\vec{u}$ and $\vec{v}$ is defined as the norm of the vector joining their endpoints. That is,

$$D(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}|.$$

Two vectors are $\vec{u}$ and $\vec{v}$ are orthogonal if and only if

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2.$$

This is just the Pythagorean theorem.

A set of vectors is an **orthogonal set** if the dot product of any pair is zero with the exception of those paired with themselves. An orthogonal set $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is linearly independent and forms an orthogonal basis for span$\{\vec{u}_1, \ldots, \vec{u}_n\}$. An **orthogonal basis** is simply an orthogonal set of basis vectors.

If $U = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is an *orthogonal* basis for some subspace, and if $\vec{x}$ is in that subspace, then we know that $\vec{x}$ can be represented as the linear combination $\vec{x} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$. The coefficients are

$$c_i = \frac{\vec{u}_i \cdot \vec{x}}{\vec{u}_i \cdot \vec{u}_i}.$$

The **orthogonal projection** of $\vec{u}$ onto a vector $\vec{v}$ is given by

$$\text{proj}_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v},$$

To write a given vector $\vec{u}$ in terms of a vector in the direction of $\vec{v}$ and a vector orthogonal to $\vec{v}$, then we write

$$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}}\, \vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}) \\ &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} + \left(\vec{u} - \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}\right). \end{aligned}$$

A set of vectors is an **orthonormal** set if it is an *orthogonal set* of *unit vectors*.

An **orthogonal matrix** is a matrix with orthonormal columns. To show that a matrix is orthogonal, just show that the columns are a set of orthonormal vectors. Also, an $m \times n$ matrix U is orthogonal if and only if $U^T U = I$.

If U is a *square* invertible and orthogonal $n \times n$ matrix, then $U^{-1} = U^T$.

If $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is an *orthogonal* basis of the subspace W which lies in $\mathbb{R}^n$, then

$$\begin{aligned} \text{proj}_W\, \vec{y} &= \text{proj}_{\vec{u}_1}\, \vec{y} + \cdots + \text{proj}_{\vec{u}_p}\, \vec{y} \\ &= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\vec{u}_p. \end{aligned}$$

The **distance** between a vector and a subspace is defined as the distance between the vector and its projection onto the subspace. So given a vector $\vec{y}$ and a subspace W,

$$D(\vec{y}, W) = |\vec{y} - \text{proj}_W\, \vec{y}|.$$

Given a non-orthogonal set of basis vectors $\{\vec{x}_1, \ldots, \vec{x}_n\}$, an orthogonal set of basis vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spanning the same space can be computed as

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\ \vdots &= \vdots \end{aligned}$$

This is the **Gram-Schmidt process**.

Given an inconsistent equation $A\vec{x} = \vec{b}$, we can find the **least squares solution** by finding the $\vec{x}$ that minimizes $||\vec{b} - A\vec{x}||$. To do that, we solve

$$(A^T A)\vec{x}_m = (A^T \vec{b}),$$

for $\vec{x}_m$, by writing it as an augmented matrix and row reducing it. The **least squares error** is the actual value of $||\vec{b} - A\vec{x}_m||$ where $\vec{x}_m$ is the vector that minimizes $||\vec{b} - A\vec{x}||$.

To get a linear least squares best fit line for a set of points, start by writing the system of equations for each of the given points.

$$\begin{aligned} b_0 + b_1 x_1 &= y_1 \\ \vdots &= \vdots \\ b_0 + b_1 x_n &= y_n. \end{aligned}$$

We can write this system of equations as the matrix equation

$$X\vec{b} = \vec{y},$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \qquad \vec{\boldsymbol{b}} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \qquad \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Since we don't have a perfect fit, there is no $\vec{\boldsymbol{b}}$ that solves $X\vec{\boldsymbol{b}} = \vec{\boldsymbol{y}}$, so we find the closest one by finding the $\vec{\boldsymbol{b}}$ that minimizes $||\vec{\boldsymbol{y}} - X\vec{\boldsymbol{b}}||$. To do that, we solve

$$(X^T X)\vec{\boldsymbol{b}}_m = (X^T \vec{\boldsymbol{y}}).$$

by setting up an augmented matrix and solving for $\vec{\boldsymbol{b}}$. Then the equation of the best fit line will be

$$y = b_0 + b_1 x.$$

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