Complex Variables
Class Notes

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Preface

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Updates

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Chapter 1

Complex Numbers

1.1 Constructing the Complex Numbers

Consider the different sets of numbers:

$$\begin{aligned} \mathbb{N} &= \text{natural numbers} \\ \mathbb{Z} &= \text{integers} \\ \mathbb{Q} &= \text{rational numbers} \\ \mathbb{R} &= \text{real numbers.} \end{aligned}$$

We define the natural numbers as $\mathbb{N} = \{1, 2, 3, \ldots\}$. We define the integers as $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. We define the rational numbers as $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$. The real numbers are not as easy to define explicitly. We cannot easily write a definition in set notation as we did with the others. They can be explicitly defined, using for example Dedekind cuts, but they are not simple like the others. Note that each set of numbers is a subset of the succeeding set. That is

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The real numbers are different from the other sets in that they form a **field**. The properties of the real numbers include all of the following, whereas the other sets do not have all these properties. For two real numbers a and b, the following properties hold:

- $a + b = b + a$, additive commutativity
- $a \cdot b = b \cdot a$, multiplicative commutativity
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, distributivity
- $a + 0 = a$, additive identity
- $a \cdot 1 = a$, multiplicative identity
- $a + (-a) = 0$, additive inverse
- $b \cdot b^{-1} = 1$, multiplicative inverse

It's the multiplicative inverse that really sets $\mathbb{R}$ apart from $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

The need for complex numbers arose from the desire to find roots/solutions for all polynomials. For example, the simple polynomial

$$x^2 + 1 = 0,$$

implies $x = \pm\sqrt{-1}$, a number that does not exist in $\mathbb{R}$, and therefore, not in any of its subsets such as $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. We can't just take the set $\mathbb{R}$ and add $\sqrt{-1}$ to it because we don't know that we won't find contradictions at some point.

When defining the complex numbers $\mathbb{C}$, we want to construct them as an extension of $\mathbb{R}$. That is, we want to have the real numbers be a subset of the complex numbers. We start by defining $\mathbb{C}$ as a two-dimensional vector space $\mathbb{R}^2$ with the operations of addition and multiplication. $\mathbb{R}^2$ is simply any ordered pair of real numbers. That is,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}.$$

We have defined a complex number as an ordered pair (a, b), which means we can represent it on a 2D cartesian plane. Since $\mathbb{R}^2$ is a vector space, unsurprisingly, complex addition is just vector addition. To add two complex numbers, you add them componentwise like

$$(a, b) + (c, d) = (a + c, b + d).$$

Since it is just vector addition, we can also do it graphically on a cartesian plane just like with vectors.

What about complex multiplication? A naive definition might be $(a, b) \cdot (c, d) = (ac, bd)$. This *is* a number system, but it is not a very useful one since it does not have the properties of a field. The actual definition of complex multiplication is

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

This fulfills the requirements of $\mathbb{C}$ being a field.

We have now defined $\mathbb{C}$ as a field under addition and multiplication. Its additive identity is $(0, 0)$ since

$$(a, b) + (0, 0) = (a, b),$$

and its multiplicative identity is $(1, 0)$ since

$$(a, b) \cdot (1, 0) = (a, b).$$

Its multiplicative inverse is

$$(a, b)^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

Every ordered pair except for $(0, 0)$ has an inverse.

A real number is just an ordered pair where the second element is zero. For example, the complex number $(2, 0)$ is the real number 2. In other words, $\mathbb{R}$ is equivalent to $\mathbb{R} \times \{0\}$, and the real number a is equivalent to the complex number $(a, 0)$. Also, since

$$\begin{aligned} (a + b) &\Leftrightarrow (a + b, 0) \Leftrightarrow (a, 0) + (b, 0) \\ (a \cdot b) &\Leftrightarrow (a \cdot b, 0) \Leftrightarrow (a, 0) \cdot (b, 0), \end{aligned}$$

this correspondence preserves addition and multiplication of $\mathbb{R}$ and shows that indeed, $\mathbb{R}$ is a subset of this number system $\mathbb{C}$ that we have defined.

Now consider an arbitrary complex number (a, b). We can write this as

$$\begin{aligned} (a, b) &= (a, 0) + (0, b) \\ &= (a, 0) + (b, 0) \cdot (0, 1) \\ &= a + b(0, 1). \end{aligned}$$

That is, the complex number (a, b) is the real number a plus the real number b times the complex number $(0, 1)$.

If we square this complex number, we get

$$(0, 1)^2 = (0, 1) \cdot (0, 1) = (-1, 0) = 1.$$

That is, the square of this complex number is the real number -1. We call this complex number i, or the **imaginary unit**

$$i^2 = -1, \qquad i = \sqrt{-1}.$$

So the number $a + b(0, 1)$ can be written as $a + bi$. This is the rectangular or cartesian form of complex numbers. We will now stop using the (a, b) convention to write complex numbers, and use $a + bi$ instead.

1.2 Cartesian Form of Complex Numbers

A **complex number** in rectangular or cartesian form is a number

$$a + bi,$$

where a and b are real numbers and i is the imaginary unit $i = \sqrt{-1}$. In other words,

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

The real part of this complex number is a and the imaginary part is b. If the real part of a complex number is zero, the number is typically called an **imaginary number**. If the imaginary part of a complex number is zero, the number is typically called a real number.

Using the fact that $i^2 = -1$ and using i as a variable makes addition and multiplication of complex numbers a lot easier. Complex addition is

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Complex multiplication is

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

This means we won't have to memorize the strange complex multiplication rule, we simply treat i as a variable.

In summary, to add two complex numbers, just add their corresponding components. To multiply two complex numbers, you "FOIL" them like with binomials.

Typically in this chapter, letters at the lower end of the alphabet, such as $a, b, c, ...$ will be real numbers and letters at the upper end such as u, v, w, and z will be complex numbers. For example, $z = a + bi$. We reserve x and y for their typical use as variables.

The **complex conjugate** of $z = a + bi$, is defined as

$$\overline{z} = \overline{a + bi} = a - bi.$$

Basically, to obtain the complex conjugate of a number, you just change the sign of every i. Geometrically, complex conjugation is a reflection of the number through the real axis.

The **modulus** or **norm** of a complex number $z = a + bi$ is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus is the complex analogue of the absolute value. Geometrically, the modulus of a complex number is just its distance from the origin when plotted on the plane. Similarly, $|z - w|$ is just the Euclidean distance between z and w on the same plane.

Note that if z and w are complex numbers, then $z < w$ is meaningless unless they are both completely real. However, $|z| < |w|$ does have meaning. It means z is closer to the origin than w.

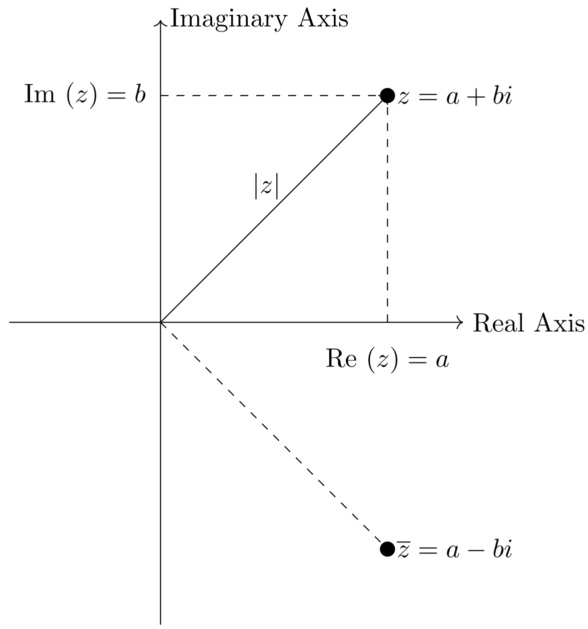
The complex number $a + bi$ can be represented as a vector ending at (a, b) on a 2-dimensional complex plane with a horizontal axis of real numbers and a vertical axis of imaginary numbers. This coordinate system designed for complex numbers is called the **Argand plane**.

Tip

When working with geometric objects such as circles, ellipses, and lines on the complex plane defined by equations such as $|z + 3 - 2i| = 4$, remember that $|z - z_1|$ is the distance between z and z_1, not the distance between z and $-z_1$. In other words, don't forget the negative sign in the formula for the distance between two complex numbers.

Tip

When doing proofs that involve $|z|$, always try to use $|z|^2$, and then take the square root in the end. This way you can replace it by $z\overline{z}$ which is usually much easier to work with.



Since complex numbers can be represented as vectors, complex addition is the same as vector addition–just add the components. You can also add complex numbers graphically on the Argand plane just as you would vectors. Complex multiplication is a little different. It is not like the vector dot product or the vector cross product. It represents an extension and a rotation of a complex number (i.e. vector in the Argand plane).

Recall that $|z - w|$ is the distance between z and w on the Argand plane. So $|z - z_0|$ where z is allowed to vary and z_0 is a constant, describes a circle in the Argand plane of radius $|z - z_0|$ centered at z_0. For example, $|z + 3 - 2i| = 4$ is a circle of radius 4 with center at $-3 + 2i$.

There is an important relationship between the modulus of a number and the complex conjugate, and it is that

$$|z|^2 = z\overline{z}.$$

That is, the square of the modulus of a complex number is equal to the complex number times its complex conjugate. This can easily be proved as

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2. \end{aligned}$$

More precisely, the product of a complex number with its complex conjugate is

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 + i[a, b],$$

where $[a, b]$ is called the **commutator** of a and b. If a and b commute, that is, if $ab - ba = 0$, then $[a, b] = 0$.

Example 1.2.1

Show that

$$|z \cdot w| = |z| \cdot |w|.$$

We start by squaring the left side so that we can work with the conjugate instead of the norm.

$$\begin{aligned} |zw|^2 &= zw \cdot \overline{zw} \\ &= zw\overline{z}\,\overline{w} \\ &= z\overline{z}w\overline{w} \\ &= |z|^2|w|^2. \end{aligned}$$

Then just take the square root of both sides to get

$$|z \cdot w| = |z| \cdot |w|.$$

Following are some important properties of the complex conjugate and modulus for complex numbers $z = a + bi$ and $w = c + di$.

$$\begin{aligned} \overline{z+w} &= \overline{z} + \overline{w} \\ \overline{\overline{z}} &= z \\ \overline{z \cdot w} &= \overline{z} \cdot \overline{w} \\ \overline{\left(\frac{z}{w}\right)} &= \frac{\overline{z}}{\overline{w}} \\ \frac{1}{z} &= \frac{\overline{z}}{|z|^2} \\ \mathrm{Re}(z) = a &= \frac{z + \overline{z}}{2} \\ \mathrm{Im}(z) = b &= \frac{z - \overline{z}}{2i} \\ |z \cdot w| &= |z| \cdot |w| \\ \left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \\ |\overline{z}| &= |z| \\ |z \pm w| &\leq |z| + |w|. \end{aligned}$$

Notice that when it comes to conjugation and the operations of addition and multiplication, you can do them in either order. That is, multiplying a pair of complex numbers then conjugating the result is the same as conjugating a pair of complex numbers and then multiplying the result. The last property in the list is the **triangle inequality**.

Example 1.2.2

Show that

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Again, we square both sides so we can work with the conjugate instead of the modulus. In the end, we just take the square root.

$$\begin{aligned} \left|\frac{z}{w}\right|^2 &= \left(\frac{z}{w}\right)\overline{\left(\frac{z}{w}\right)} \\ &= \left(\frac{z}{w}\right)\left(\frac{\overline{z}}{\overline{w}}\right) \\ &= \frac{z\overline{z}}{w\overline{w}} \\ &= \frac{|z|^2}{|w|^2}. \end{aligned}$$

Tip

When proving a lot of basic properties about complex numbers such as z and w, it often helps to use the component form $z = a + bi$ and $w = c + di$.

Now we just take the square root of both sides.

Example 1.2.3

Show that
$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

Just write both sides in cartesian form and show that they simplify to the same thing.

$$\begin{aligned}
\overline{z \cdot w} &= \overline{(a+bi)(c+di)} \\
&= \overline{(ac-bd)+(ad+bc)i} \\
&= (ac-bd)-(ad+bc)i \\
\overline{z} \cdot \overline{w} &= \overline{(a+bi)} \cdot \overline{(c+di)} \\
&= (a-bi)\cdot(c-di) \\
&= (ac-bd)-(ad+bc)i.
\end{aligned}$$

To divide a complex number by another complex number, we use the property

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

This property is the result of multiplying the left side by 1 and simplifying as shown here

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} \\
&= \frac{\overline{z}}{z\overline{z}} \\
&= \frac{\overline{z}}{|z|^2}.
\end{aligned}$$

So to divide a complex number by another, we multiply the top and bottom by the conjugate of the bottom and simplify.

Notice that

$$\frac{1}{i} = -i,$$

which can be shown by multiplying the top and bottom of the left side by i.

Example 1.2.4

If $z = 2 - 3i$, find its multiplicative inverse $\frac{1}{z}$.
To do this, we just multiply the top and bottom by the conjugate of z to get

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2-3i} \\
&= \frac{1}{2-3i} \cdot \frac{2+3i}{2+3i} \\
&= \frac{2+3i}{13} \\
&= \frac{2}{13} + \frac{3}{13}i.
\end{aligned}$$

Example 1.2.5

Plot the set of points in the complex plane defined by

$$|3\overline{z} + 6i| = 5.$$

To get this in the standard form of a circle, we have to start by factoring out a 3, using the property that $|zw| = |z||w|$

$$\begin{aligned} 3|\overline{z} + 2i| &= 5 \\ |\overline{z} + 2i| &= \frac{5}{3}. \end{aligned}$$

Next, we use the fact that $|\overline{z} + 2i| = |\overline{z - 2i}|$ to write

$$|\overline{z - 2i}| = \frac{5}{3}.$$

Finally, we use the fact that $|\overline{z}| = |z|$ to write

$$|z - 2i| = \frac{5}{3}.$$

This is now in the standard form of the equation of a circle with radius $\frac{5}{3}$ and center at $2i$.

Example 1.2.6

If $z = 3 + 2i$ and $w = -1 + i$, find, $z + w$, zw, and $\frac{z}{w}$.

$$\begin{aligned} z + w &= 3 + 2i - 1 + i = 2 + 3i \\ zw &= (3 + 2i)(-1 + i) = -5 + i \\ \frac{z}{w} &= \frac{3 + 2i}{-1 + i} \cdot \frac{-1 - i}{-1 - i} = -\frac{1}{2} - \frac{5}{2}i. \end{aligned}$$

1.3 Triangle Inequality

The **triangle inequality** can be written as

$$|z + w| \leq |z| + |w|.$$

The proof of the triangle inequality is as follows

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} \\ &= (z + w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \end{aligned}$$

Now, we make use of the fact that $w = \overline{\overline{w}}$ to write $w\overline{z}$ as $\overline{z\overline{w}}$.

$$|z + w|^2 = |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2.$$

However, notice that $z\overline{w} + \overline{z\overline{w}}$ is $z\overline{w}$ plus its complex conjugate. From the property $\text{Re}(z) = a = \frac{z+\overline{z}}{2}$ we know that $z\overline{w} + \overline{z\overline{w}} = 2\text{Re}(z\overline{w})$, so we have that

$$|z + w|^2 = |z|^2 + 2\text{Re}(z\overline{w}) + |w|^2.$$

Tip

Keep in mind when working with inequalities that if $A \in \mathbb{R}$ then $|A| = A$ or $-A$ depending on the sign of A.

Since the real part of a complex number is less than or equal to its modulus, we know that $\text{Re}(z\overline{w}) \leq |z\overline{w}|$, so

$$|z+w|^2 \leq |z|^2 + |w|^2 + 2|z\overline{w}|.$$

From the geometric interpretation of $|z|$ (i.e. the Argand plane) we can easily verify that the real part of a complex number is less than or equal to its modulus. Next, we note that $|z\overline{w}| = |z| \cdot |\overline{w}|$ and that $|\overline{w}| = |w|$, so we get

$$|z+w|^2 \leq |z|^2 + |w|^2 + 2|z| \cdot |w|.$$

Now, the right side is just a square

$$|z+w|^2 \leq (|z| + |w|)^2.$$

Taking the square root of both sides gives us the triangle inequality. Geometrically, the triangle inequality is the statement that one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We also find that

$$|z-w| = |z + (-w)|,$$

which by the triangle inequality, gives us

$$|z+(-w)| \leq |z| + |-w| = |z| + |-1||w| = |z| + |w|.$$

So we can actually write the triangle inequality as

$$|z \pm w| \leq |z| + |w|.$$

We can also show that

$$\big||z| - |w|\big| \leq |z-w|,$$

by using the useful trick of adding and subtracting w. Notice that $|z| = |(z-w)+w|$. By the triangle inequality, then $|(z-w)+w| \leq |z-w| + |w|$. Subtracting $|w|$ from both sides gives us

$$|z| - |w| \leq |z-w|.$$

Similarly, by adding and subtracting z from w then using the triangle inequality as above, we can show that

$$|w| - |z| \leq |w-z|.$$

But $|w-z| = |-(z-w)| = |-1||z-w| = |z-w|$, so we actually have that

$$\pm|z| \mp |w| \leq |z-w|.$$

1.4 Polar Form of Complex Numbers

Recall the Maclaurin expansions

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},
\end{aligned}$$

where $x \in \mathbb{R}$. What happens if we replace x with a complex number?

Definition:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}.$$

If we plug in $z = i\theta$ where $\theta \in \mathbb{R}$ then

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}.$$

Separating the even and odd terms of the series gives us

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}.$$

Using the fact that $(i\theta)^{2n} = (i^2\theta^2)^n = (-1)^n\theta^{2n}$ and $(i\theta)^{2n+1} = (i\theta)^{2n}(i\theta) = (-1)^n\theta^{2n}(i\theta) = i(-1)^n\theta^{2n+1}$, we can write it as

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(-1)^n\theta^{2n+1}}{(2n+1)!}.$$

But by the Maclaurin series given earlier, this is just

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This result is known as **Euler's formula**.

By letting $\theta = \pi$ in Euler's formula, we get **Euler's identity**

$$e^{i\pi} + 1 = 0.$$

Notice that we have the five most important numbers in mathematics in one equation. Euler's identity brings together number theory (0 and 1), analysis (e), geometry (π), and complex analysis (i).

Euler's formula gives us a new way to write complex numbers—the polar form. If $z = a + bi$, the polar form is $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$. So

$$z = |z|e^{i\,arg(z)}.$$

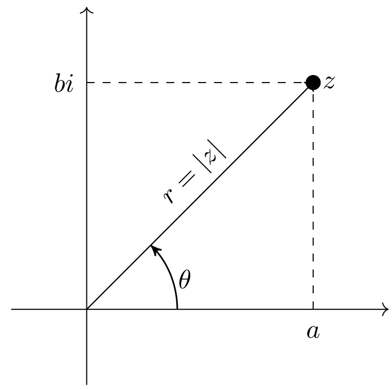
The notation $\arg z$ means the **argument** of z, and it refers to the angle that z makes with the positive real axis on the Argand plane. We can also just write

$$z = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

where

$$\begin{aligned} r &= |z| = \sqrt{a^2+b^2} \\ \theta &= \arg z = \tan^{-1}\frac{b}{a} \\ a &= r\cos\theta \\ b &= r\sin\theta. \end{aligned}$$

We can use these equations to convert back and forth between the cartesian $z = a + bi$ and polar $z = re^{i\theta}$ forms of a complex number.



Notice that if we rotate a complex number by 2π, we end up with the same complex number. Therefore, arg z can have infinite values. We call the one between $-\pi$ and π the **principal argument** and denote it with an uppercase letter as in Arg z. That is, the principle argument is the smallest positive angle that a complex number makes with the positive real axis.

Since a complex number can be plotted as a point on the Argand diagram, for $z = a + bi$, we can write

$$\begin{aligned} a &= |z|\cos\theta \\ b &= |z|\sin\theta \end{aligned}$$

Therefore, the complex number can be written as

$$\begin{aligned} z &= a + bi \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|(\cos\theta + i\sin\theta). \end{aligned}$$

Note that another common way of writing complex numbers is as

$$z = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta.$$

To convert from polar form to standard form on a TI-84 calculator, you can use **P ▶ Rx**(r, θ) from the **ANGLE** button to get the x value of the complex number and use **P ▶ Ry**(r, θ) to get the y value.

Similarly, to convert from standard form to polar form on a TI-84 calculator, you can use **R ▶ Pr**(x, y) from the **ANGLE** button to get the r value of the complex number and use **R ▶ P**$\theta(x, y)$ to get the angle.

We will now look at the different operations on complex numbers in polar form. The modulus gives us

$$\begin{aligned} |z| &= \left|re^{i\theta}\right| \\ &= |r|\left|e^{i\theta}\right| \\ &= |r|\sqrt{(\cos\theta)^2 + (i\sin\theta)^2} \\ &= |r|. \end{aligned}$$

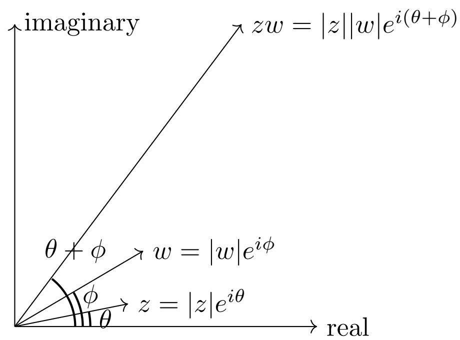
Conjugation gives us

$$\begin{aligned} \overline{z} &= \overline{re^{i\theta}} \\ &= \overline{r}\overline{e^{i\theta}} \\ &= r\overline{(\cos\theta + i\sin\theta)} \\ &= r(\cos\theta - i\sin\theta) \\ &= r(\cos(-\theta) + i\sin(-\theta)) \\ &= re^{-i\theta}. \end{aligned}$$

So again conjugation amounts to just changing the sign on any i. Note that $\overline{r} = r$ since r is a real number. For multiplication, we have to use a trigonometric identity. If $z = re^{i\theta}$ and $w = se^{i\phi}$, then

$$\begin{aligned} zw &= \left(re^{i\theta}\right)\left(se^{i\phi}\right) \\ &= rs\left(\cos\theta + i\sin\theta\right)\left(\cos\phi + i\sin\phi\right) \\ &= rs(\cos\theta\cos\phi + i\cos\theta\sin\phi \\ &\qquad + i\sin\theta\cos\phi - \sin\theta\sin\phi) \\ &= rs(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &\qquad + irs(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\ &= rse^{i(\theta+\phi)}. \end{aligned}$$

This demonstrates that to multiply two complex numbers in polar form, we just multiply their magnitudes and add their arguments. Geometrically, it shows that complex multiplication is a stretching (the rs part) and a rotation (the $\theta + \phi$ part) on the complex plane.



We used a trigonometry identity to show complex multiplication in polar form. However, since $e^u e^v = e^{u+v}$ also holds true for all complex numbers, we could just as easily have proved the trigonometry identity using complex multiplication. In fact, almost all trigonometry identities can be proved by multiplying complex numbers.

1.5 Proving Trigonometry Formulas

Double Angle Formulas

From Euler's formula, we know that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Squaring both sides gives us

$$\begin{aligned} \left(e^{i\theta}\right)^2 &= (\cos\theta + i\sin\theta)^2 \\ e^{i2\theta} &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta, \end{aligned}$$

but notice that we could also just have substituted 2θ for θ in the original formula to get

$$e^{i(2\theta)} = \cos(2\theta) + i\sin(2\theta).$$

So we have that

$$\cos(2\theta) + i\sin(2\theta) = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta. \tag{1.1}$$

If we substitute $-\theta$ for θ in Euler's formula, we get

$$e^{-i\theta} = \cos\theta - i\sin\theta,$$

since cosine is an even function and sine is an odd function. Substituting in -2θ for $-\theta$ gives us

$$e^{i(-2\theta)} = \cos(2\theta) - i\sin(2\theta),$$

whereas squaring it gives us

$$\left(e^{i(-\theta)}\right)^2 = \cos^2\theta - 2i\sin\theta\cos\theta - \sin^2\theta.$$

Combining the two gives us

$$\cos(2\theta) - i\sin(2\theta) = \cos^2\theta - 2i\sin\theta\cos\theta - \sin^2\theta. \tag{1.2}$$

Subtracting Eq. 1.2 from Eq. 1.1 gives us the familiar trigonometry formula

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

Adding Eq. 1.2 and Eq. 1.1 gives us the familiar trigonometry formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Using the same process, we can substitute 3θ and -3θ to get

$$\begin{aligned} e^{i(3\theta)} &= \cos(3\theta) + i\sin(3\theta) \\ e^{i(-3\theta)} &= \cos(3\theta) - i\sin(3\theta), \end{aligned}$$

and we can also cube both sides of Euler's equation to get

$$\begin{aligned} \left(e^{i\theta}\right)^3 &= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta \\ \left(e^{-i\theta}\right)^3 &= \cos^3\theta - 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta + i\sin^3\theta. \end{aligned}$$

Equating the relevant pairs and adding one to the other and subtracting one from the other gives us the trigonometry formulas

$$\begin{aligned} \sin 3\theta &= -\sin^3\theta + 3\sin\theta\cos^2\theta \\ \cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta. \end{aligned}$$

Sum and Difference Formulas

The following is a quick derivation of the sum and difference formulas that you can perform in less than a minute if you're having trouble remembering them. Start with

$$e^{i(a+b)} = e^{ia}e^{ib}.$$

Next, expand both sides using Euler's formula to get

$$\begin{aligned} \cos(a+b) + i\sin(a+b) &= (\cos a + i\sin a)(\cos b + i\sin b) \\ &= \cos a\cos b + i\cos a\sin b \\ &+ i\sin a\cos b - \sin a\sin b. \end{aligned}$$

Finally, just equate the real and imaginary parts of both sides to get

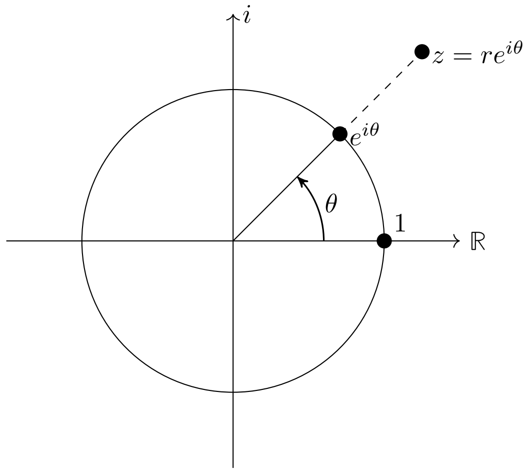
$$\begin{aligned} \cos(a+b) &= \cos a\cos b - \sin a\sin b \\ \sin(a+b) &= \sin a\cos b + \cos a\sin b. \end{aligned}$$

To find the negative versions, just substitute $-b$ for b in the above two formulas and use the fact that cosine is an even function and sine is an odd function to get

$$\begin{aligned} \cos(a-b) &= \cos a\cos b + \sin a\sin b \\ \sin(a-b) &= \sin a\cos b - \cos a\sin b. \end{aligned}$$

1.6 Roots of Complex Numbers

Recall that the complex number $e^{i\theta}$ falls on the unit circle (i.e. circle of radius 1) on the complex plane.



Tip

Remember that $-\pi < \text{Arg } z \leq \pi$. So instead of $\frac{5\pi}{4}$, you would use $-\frac{3\pi}{4}$. Instead of using $-\pi$, you would use π instead.

To locate a number $z = re^{i\theta}$ on the complex plane, we just go up an angle θ from the positive real axis, and out a distance r from the origin. Here, $r = |z|$ and $\theta = \text{Arg } z$ where $\theta = \text{Arg } z$ means $-\pi < \theta \leq \pi$. Notice that π and $-\pi$ are the same angle, but for the principal argument, we use only π.

What does it mean to take a root of a complex number? If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\sqrt[n]{z} = z^{\frac{1}{n}} = w \Leftrightarrow w^n = z.$$

That is, the nth root of a complex number z is the number w such that w raised to the nth power gives z.

Every complex number (except zero) has n distinct nth roots. This is not true for real roots of real numbers. For example, the only distinct real cube root of 1 is 1. However, $z^3 = 1$ has three complex roots if $z \in \mathbb{C}$.

The **nth roots of unity** are the nth roots of the number 1. That is, they are the complex numbers w such that $w^n = 1$. The nth roots of unity are

$$w^k = e^{\frac{2\pi k}{n} i}, \qquad k = 0, 1, 2, \ldots, n-1.$$

Alternatively, we can say the nth roots of unity are

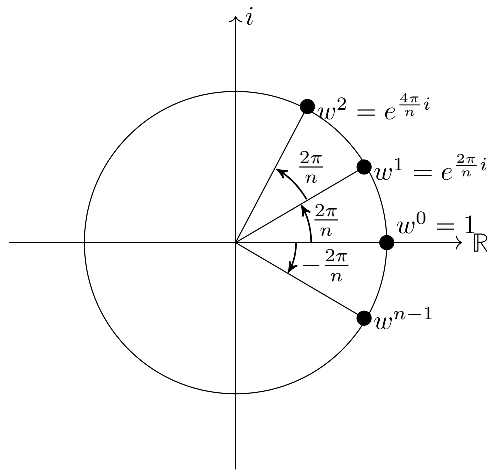
$$w^0, w^1, w^2, \ldots, w^{n-1}, \text{ where } w = e^{\frac{2\pi}{n} i}.$$

To demonstrate that these are all roots of unity, we have to show that each w^k, when raised to the nth power, gives 1.

$$\begin{aligned} w^k &= e^{\frac{2\pi k}{n} i} \\ (w^k)^n &= \left(e^{\frac{2\pi k}{n} i}\right)^n \\ &= e^{2\pi k i} \\ &= \cos(2\pi k) + i \sin(2\pi k) \\ &= 1. \end{aligned}$$

The first one, $w^0 = 1$ is called the **principal root** of 1. The w is called the **generator**.

Geometrically, the nth roots of unity all fall on the unit circle. The principal root of 1 is always just one, so on the complex plane, the first root is always at $(1, 0)$ on the complex plane. The second root of unity is $w^1 = w = e^{\frac{2\pi}{n}i}$. Notice from the argument in the exponent that the angle of w is $\frac{1}{n}$ of the way around the unit circle. Similary, the third root is another $\frac{1}{n}$ around the unit circle. That is, the nth roots of unity are evenly spaced around the unit circle with the first one at $(1, 0)$.



Example 1.6.1

Find the cube roots of unity.
Our generating function is

$$w^k = e^{\frac{2\pi k}{3}i}, \qquad k = 0, 1, 2.$$

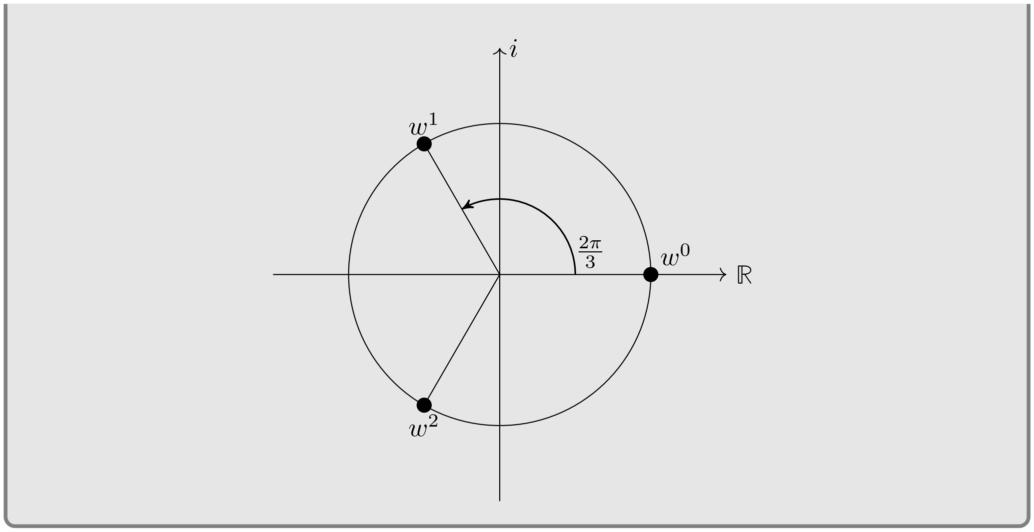
The first one is just $w^0 = 1$. The other two are found as follows

$$\begin{aligned} w^1 &= e^{\frac{2\pi}{3}i} \\ &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ w^2 &= e^{\frac{4\pi}{3}i} \\ &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

So the three cube roots of 1 are

$$1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Plotting them gives us te following picture. Notice that each of the three roots are evenly spaced around the unit circle.



We want to be able to find the nth roots of arbitrary complex numbers. So far we know how to find the roots of 1. If $z \neq 0$, then

$$z = re^{i\theta}, \qquad |z| = r > 0,\ \theta = \operatorname{Arg} z,\ -\pi < \theta \leq \pi.$$

Zero is a special case. If $z = 0$ then there is only one nth root, namely, 0. We want to find w such that $w^n = z$. Let $r^{\frac{1}{n}}$ be the positive nth root of the real number r, then

$$z = re^{i\theta} = w^n = \left(r^{\frac{1}{n}} e^{\frac{\theta}{n} i}\right)^n.$$

Then we call

$$w = r^{\frac{1}{n}} e^{\frac{\theta}{n} i},$$

the principal nth root of z. Remember that θ is the principal argument. Note that

$$z = (w \cdot u)^n = w^n u^n = z \cdot u^n,$$

implies that $u^n = 1$, that is, u is an nth root of unity. This tells us that the nth roots of z are

$$w, wu, wu^2, \ldots, wu^{n-1},$$

where

$$u = e^{\frac{2\pi}{n} i},$$

are the roots of unity.

So to find the nth roots of a complex number $z = re^{i\theta}$,

1. Draw the complex plane and locate z in it.
2. Convert z to polar form if it is not already.
3. Find the principal nth root of z, which is

$$w = r^{\frac{1}{n}} e^{\frac{\theta}{n} i}.$$

4. Find the nth roots of unity

$$u^k = e^{\frac{2\pi k}{n} i}, \qquad k = 0, 1, 2, \ldots, n-1.$$

5. Then the nth roots of z are

$$w, wu, wu^2, \ldots, wu^{n-1}.$$

6. If necessary, convert the roots back to cartesian form using Euler's formula.

To locate the nth roots of $z = re^{i\theta}$ geometrically,

1. Plot the nth roots of unity on the unit circle.
2. Multiply every point by $r^{\frac{1}{n}}$ to expand the circle from a unit circle to the radius on which the roots of z lie.
3. Rotate the whole system counterclockwise by the angle $\frac{\theta}{n}$.

Raising both sides of Euler's formula to the nth power gives us

$$\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n .$$

Notice that on the left side, $\left(e^{i\theta}\right)^n = e^{i(n\theta)}$, so making a simple substitution $\theta = nx$ in Euler's formula gives us **De Moivre's theorem**

$$(\cos x + i\sin x)^n = e^{inx} = \cos(nx) + i\sin(nx).$$

If we replace θ with $-\theta$ in Euler's formula, we get

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

since cosine is an even function and sine is an odd function. By adding this to Euler's formula, we get

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

Similarly, by subtracting it from Euler's formula, we get

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

Alternative Method

An alternative method of multiplying and dividing complex numbers and finding their roots is detailed below. This method starts from Euler's formula and De Moivre's formula instead of the exponential form.

To multiply two complex numbers in polar form, multiply their moduli and add their arguments:

$$(r_1 \text{ cis } \theta_1) \cdot (r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2).$$

Complex multiplication is easily visualized on the complex plane. The magnitude of the product of two complex numbers is the magnitudes of the two numbers multiplied together. The angle of the product of two complex numbers is the sum of the angles of the inputs.

De Moivre's Theorem is a useful formula for finding roots and powers of complex numbers. It states that

$$\begin{aligned} z^n &= r^n \text{ cis } (n\theta) \\ [r(\cos\theta + i\sin\theta)]^n &= r^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

Example 1.6.2

Simplify

$$\left(\sqrt{2} \text{ cis } \frac{7\pi}{24}\right)^8 .$$

Using De Moivre's theorem, we get

$$\begin{aligned}\left(\sqrt{2}\text{ cis }\frac{7\pi}{24}\right)^8 &= \left(\sqrt{2}\right)^8 \text{ cis } 8\cdot\frac{7\pi}{24}\\ &= 16 \text{ cis } \frac{7\pi}{3}\\ &= 16 \text{ cis } \frac{\pi}{3}.\end{aligned}$$

To divide two complex numbers in trigonometric form, divide their moduli and subtract their arguments

$$\frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2).$$

Every complex number has exactly n distinct nth roots. To find the n, nth roots of a complex number

$$z^{\frac{1}{n}} = \text{ cis } \left(\frac{\theta}{n} + \frac{360°}{n}k\right), \text{ where } k = 0, 1, 2, \ldots n-1.$$

The root corresponding to $k = 1$ is called the principal nth root of the complex number. Using a calculator will only give the principal nth root of a complex number.

Example 1.6.3

Find the three cube roots of $32\sqrt{3} + 32i$.

Converting to trig form gives us $32\sqrt{3} + 32i = 64 \text{ cis } 30°$. Then using the formula given above, we have that $n = 3$ for cube roots, so the roots are

$$\begin{aligned}4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[0]\right) &= 4 \text{ cis } 10°\\ 4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[1]\right) &= 4 \text{ cis } 130°\\ 4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[2]\right) &= 4 \text{ cis } 250°\end{aligned}$$

Note: The n, nth roots in trigonometric form are distributed evenly (graphically). In other words, the 4, 4th roots of a complex number will have displayed graphically as 4 vectors that are 90 degrees apart.

1.7 Summary: Complex Numbers

Cartesian Form

A **complex number** can be written in cartesian form as

$$a + bi.$$

To add two complex numbers, just add their corresponding components. To multiply two complex numbers, you "FOIL" them like with binomials. To divide two complex numbers, multiply the top and bottom by the conjugate of the denominator and simplify.

To take the complex conjugate of a number, you just change the sign of every i. Geometrically, complex conjugation is a reflection of the number through the real axis.

The **modulus** or **norm** of a complex number $z = a + bi$ is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus is the complex analogue of the absolute value. Geometrically, the modulus is just its distance from the origin when plotted on the plane. The quantity $|z - z_0|$ where z is allowed to vary and z_0 is a constant, describes a circle in the Argand plane of radius $|z - z_0|$ centered at z_0.

An important relationship is

$$|z|^2 = z\overline{z}.$$

When doing proofs that involve $|z|$, always try to use $|z|^2$, and then take the square root in the end. This way you can replace it by $z\overline{z}$ which is usually much easier to work with.

Following are some important properties of the complex conjugate and modulus for complex numbers $z = a + bi$ and $w = c + di$.

$$\begin{aligned}
\overline{z + w} &= \overline{z} + \overline{w} \\
\overline{\overline{z}} &= z \\
\overline{z \cdot w} &= \overline{z} \cdot \overline{w} \\
\overline{\left(\frac{z}{w}\right)} &= \frac{\overline{z}}{\overline{w}} \\
\frac{1}{z} &= \frac{\overline{z}}{|z|^2} \\
\mathrm{Re}(z) = a &= \frac{z + \overline{z}}{2} \\
\mathrm{Im}(z) = b &= \frac{z - \overline{z}}{2i} \\
|z \cdot w| &= |z| \cdot |w| \\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \\
|\overline{z}| &= |z| \\
|z \pm w| &\leq |z| + |w|.
\end{aligned}$$

Polar Form

Euler's formula is

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Raising both sides to the nth power and making a simple substitution gives us **De Moivre's theorem**

$$(\cos x + i \sin x)^n = e^{inx} = \cos(nx) + i \sin(nx).$$

To write a complex number in polar form, we write

$$z = a + bi = re^{i\theta} = r(\cos\theta + i \sin\theta),$$

where

$$\begin{aligned}
r &= |z| = \sqrt{a^2 + b^2} \\
\theta &= \arg z = \tan^{-1}\frac{b}{a} \\
a &= r\cos\theta \\
b &= r\sin\theta.
\end{aligned}$$

We can use these transformation equations to convert back and forth between the Cartesian and polar forms of a complex number. We can also find the polar form of a complex number by plotting the number on the complex plane, and using the Pythagorean theorem to find r. We are often given complex numbers that form some common triangle on the complex plane, so it is easy to find the angle θ as well.

If we rotate a complex number by 2π, we end up with the same complex number. Therefore, $\arg z$ can have infinite values. We call the one between $-\pi$ and π the **principal argument** and denote it with an uppercase letter as in Arg z. That is, the principle argument is the smallest positive angle that a complex number makes with the positive real axis.

To locate a number $z = re^{i\theta}$ on the complex plane, we just go up an angle θ from the positive real axis, and out a distance r from the origin.

To multiply two complex numbers in polar form, we just multiply their magnitudes and add their arguments. Geometrically, it shows that complex multiplication is a stretching and a rotation on the complex plane.

The **nth roots of unity** are

$$1^{\frac{1}{n}} = e^{\frac{2\pi k}{n}i}, \qquad k = 0, 1, 2, \ldots, n-1.$$

Geometrically, the nth roots are evenly spaced around the unit circle with the first one at $(1, 0)$.

The nth roots of a complex number $z = re^{i\theta}$ are given by

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}, \qquad k = 0, 1, 2, \ldots, n-1.$$

Notice that you have to convert to polar form to find the roots. If necessary convert back to Cartesian form in the end by using Euler's formula. To locate the nth roots of $z = re^{i\theta}$ geometrically,

1. Plot the nth roots of unity on the unit circle.
2. Multiply every point by $r^{\frac{1}{n}}$ to expand the circle from a unit circle to the radius on which the roots of z lie.
3. Rotate the whole system counterclockwise by the angle $\frac{\theta}{n}$.

To raise a complex number to a power, we can expand it out algebraically if the power is small—2 or 3. For higher powers, it is faster to convert the number to polar form, raise to the power, then convert it back to Cartesian form using Euler's formula.

Chapter 2

Analytic Functions

Complex analysis, that is, calculus with complex functions, was initiated and developed by Karl Weierstrass, Augustin Louis Cauchy, and Bernhard Riemann.

2.1 Regions of the Complex Plane

Consider a region U of the complex plane (or a set U of complex numbers). Now consider a disk of radius ε centered at the complex number z. We denote this disk as $D_\varepsilon(z)$. The region enclosed by this disk is called the **epsilon neighborhood of** z, and is denoted

$$N_\varepsilon(z) = \{w : |w - z| < \varepsilon\}.$$

Notice that $|w - z| < \varepsilon$ means the disk (not including the boundary) of radius ε centered at z. If the boundary of the disk is included, then it is called the **closed epsilon neighborhood of** z and is denoted

$$\overline{N}_\varepsilon(z) = \{w : |w - z| \leq \varepsilon\}.$$

If the center point z is excluded, then it is called the **deleted epsilon neighborhood of** z and is denoted

$$\dot{N}_\varepsilon(z) = \{w : 0 < |w - z| < \varepsilon\}.$$

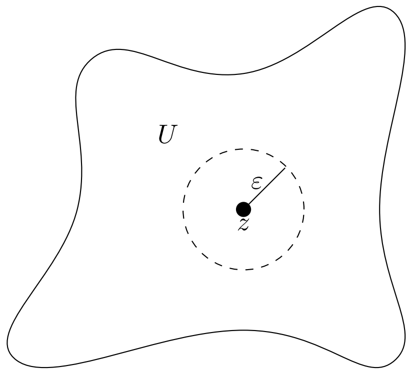
The point z is an **interior point** of U if $\exists\ \varepsilon > 0$ such that $N_\varepsilon(z) \subset U$. In other words, z is a an interior point of U if there exists some disk of non-zero radius around z that is completely contained in U.

The point z is an **exterior point** of U if $\exists\ \varepsilon > 0$ such that $N_\varepsilon(z) \cap U = \phi$. In other words, z is an exterior point of U if there exists a disk of nonzero radius around z such that the intersection of the disk and U is the null set. That is, the disk is completely outside of U.

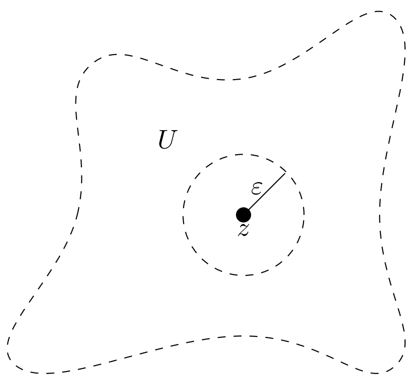
The point z is a **boundary point** of U if $\forall\ \varepsilon > 0$, $N_\varepsilon(z)$ contains points in U and points not in U. In other words, z is a boundary point of U if every possible disk around z contains points in U and points that are not in U.

U is a **closed set** if it contains all of its boundary points. For example, the set denoted by $|z| \leq 1$ is the disk of radius 1 centered at the origin. Because of the less than or equal sign, the boundary is included, so this is a closed set. Below is depicted an arbitrary closed set containing a point z and an example epsilon neighborhood of z.

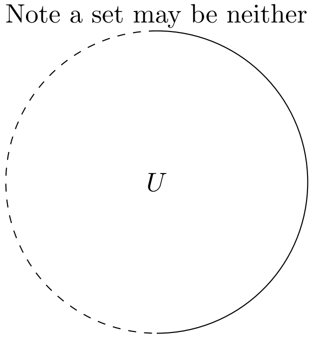
The **closure** of a set U is the closed set of all points in U and its boundary.



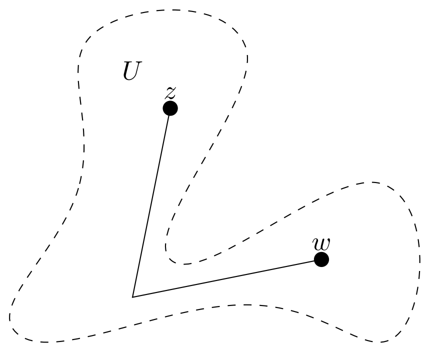
U is an **open set** if every $z \in U$ is an interior point of U. In other words, U contains no boundary points. An example of an open set is $|z| < 1$ which describes a disk of radius 1 centered at the origin not including the boundary of the disk. Incidentally, the boundary of a region U is typically denoted ∂U. Open sets are drawn with dashed boundaries as illustrated below.



Note a set may be neither open nor closed as shown in the example below.



U is a **connected space** if $\forall\ z, w \in U$, $\exists$ a polygonal path *in* U from z to w. In other words, a region is *connected* if for any pair of points in the region, there exists a polygonal (i.e. piecewise linear curve) completely within U that goes from one point to the other.



A set U is **bounded** if every point in U lies inside some circle. Otherwise, it is **unbounded**.

A **domain** is a set $U \in \mathbb{C}$ that is open and connected.

2.2 Basic Mappings

Calculus is the study of real-valued functions of a real variable. We denote such a function f by

$$f : \mathbb{R} \to \mathbb{R}.$$

For example, $f(x) = x^2$ is such a function because it is a real-valued function x^2 of a real variable x. A second example is $g(x) = \sqrt{x}$, which is denoted

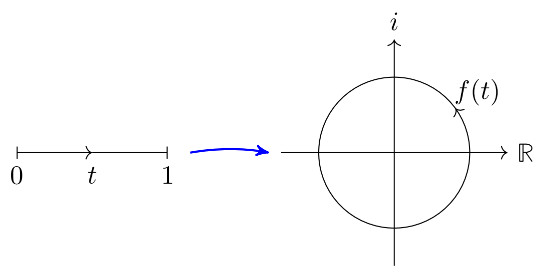
$$g : [0, \infty) \to \mathbb{R},$$

to denote that the variable takes on values in the domain $[0, \infty)$ and the function gives values in $\mathbb{R}$. This kind of function is typically visualized on a two-dimensional cartesian plane where the horizontal axis is the variable, and the vertical axis is the function value.

A second type of function is the complex-valued function of a real variable. One example is

$$f : [0, 1] \to \mathbb{C}, \qquad f(t) = e^{2\pi i t}.$$

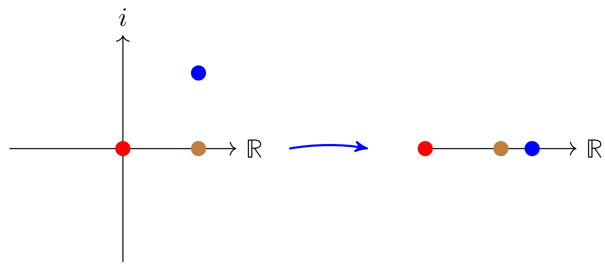
For such a function, we often think of the domain as time. In this case, t goes from 0 to 1, while $f(t)$ traces out a path on the complex plane. This kind of function shows up a lot, and we can represent such a function by a pair of graphs as shown below.



A third type of function is the real-valued function of a complex variable. An example of such a function is

$$f : \mathbb{C} \to \mathbb{R}, \qquad f(z) = |z|,$$

since it takes in complex values and returns real values. The domain is two-dimensional since it is all complex variables. This kind of function does not show up a lot, but we can visualize it by a pair of graphs as shown below.



Notice that corresponding points, the variable on the left and the function value on the right, are of the same color.

Finally, there are complex-valued functions of a complex variable. An example is

$$f : \mathbb{C} \to \mathbb{C}, \qquad f(z) = z^2.$$

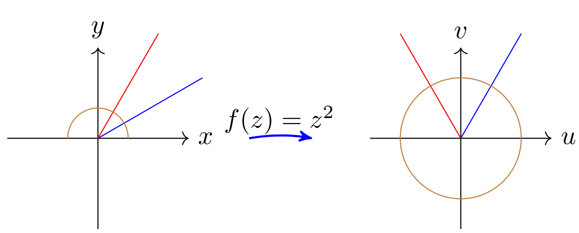
This function is the complex extension of $f(x) = x^2$, which means plugging in real numbers yields the real and expected results. This kind of function is what complex variables is primarily focused on.

Notice that with complex-valued functions of a complex variable, the input is two-dimensional (i.e. the complex numbers), and the output is two-dimensional (again the complex numbers). Since we cannot visualize 4-dimensional graphs, we have to use a pair of 2-dimensional planes to "graph" such functions.

Squaring Function

Consider the function $f(z) = z^2$. The polar form is given by $f(re^{i\theta}) = r^2e^{2i\theta}$. Notice that as θ goes from 0 to some θ_1, that the function goes around twice as far to $2\theta_1$. As θ goes from 0 to 2π, 2θ goes from 0 to 4π. For any complex number z with some distance from the origina and at some angle, that is plugged into the function, the output is a number at twice the angle and the distance squared. So if one's domain is a semi-circle of radius r, the output is a full circle with radius r^2.

To graph such functions, we can use a pair of Cartesian planes. On the left, we have the domain, and on the right we have the range. We let $z = x + iy$ and $f(z) = u + iv$, which gives us the axis labels to use.



2.3 Complex Limits

To compute the limit of a real number at a point x_0, we only have to check the limit from the left and the limit from the right. If the two agree, then the limit exists and has that value. If the two disagree, then the limit does not exist. The key point is that there are only two directions from which to approach a point.

For complex limits, there are infinite directions to approach a complex number z_0 since you are in the complex plane instead of on the real number line. For a complex limit to exist, the limit as you approach z_0 along every possible path, must be the same. That is,

$$\lim_{z \to z_0} f(z) = L, \text{ iff } \lim_{\substack{z \to z_0 \\ z \in C}} f(z) = L,$$

for all possible curves C. If any two curves do not agree on the limit, then the limit does not exist.

Example 2.3.1

Show that

$$\lim_{z\to 0} \frac{\text{Re}(z)\text{Im}(z)}{|z|^2},$$

does not exist.

First, we let $z = x + iy$ then $\text{Re}(z) = x$ and $\text{Im}(z) = y$, so our limit becomes

$$\lim_{x+iy\to 0} \frac{xy}{x^2+y^2},$$

Next, we check the limit as we approach 0 along the real axis. The key is that along this curve, $y = 0$, so the limit is

$$\lim_{x\to 0} \frac{x \cdot 0}{x^2+(0)^2} = 0.$$

Next, we check the limit as we approach 0 along the imaginary axis. Here, $x = 0$, so the limit is

$$\lim_{y\to 0} \frac{0 \cdot y}{(0)^2+y^2} = 0.$$

Finally, we check the limit as we approach 0 along the line $y = x$. Here, if we change all our y's to x's, our limit becomes

$$\lim_{x+iy\to 0} \frac{x^2}{x^2+x^2} = \lim_{x+iy\to 0} \frac{1}{2} = \frac{1}{2}.$$

This limit does not agree with the other two, so overall, the limit does not exist.

In order for a limit approaching zero to exist, all radial limits must agree. A radial limit is just the limit as you approach zero along some straight line. However, this is not a sufficient condition because all the radial limits might agree, but a curving approach could yield a different limit. Consider the limit

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} f(x,y) = \lim_{z\to 0} \frac{x^2 y}{2x^4+y^2},$$

where $z = x + iy$. Approaching zero along the real axis, we have that $y = 0$, so the limit is

$$\lim_{x\to 0} \frac{x^2 \cdot 0}{2x^4+(0)^2} = 0.$$

Approaching zero along the imaginary axis, we have that $x = 0$, so the limit is

$$\lim_{y\to 0} \frac{0 \cdot y}{2(0)^4+y^2} = 0.$$

We now approach 0 along a radial path with an arbitrary slope m. That is, it could be any radial path. Since it is a line going through $(0, 0)$ with slope m, its equation is $y = mx$. Making this substitution gives us

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2(mx)}{2x^4+(mx)^2} &= \lim_{x\to 0} \frac{mx^3}{2x^4+m^2x^2} \\
&= \lim_{x\to 0} \frac{mx}{2x^2+m^2} \\
&= 0.
\end{aligned}$$

This shows that the limit is 0 along every radial path. However, what happens if we now approach along the parabolic path $y = x^2$? Substituting this in gives us

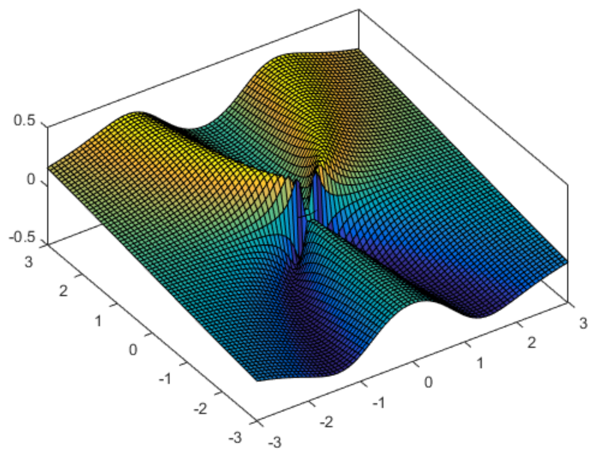
$$\lim_{z\to 0} \frac{x^2(x^2)}{2x^4 + (x^2)^2} = \lim_{z\to 0} \frac{x^4}{3x^4} = \frac{1}{3}.$$

So the limit does not exist even though it is zero along every radial approach.

A graph of

$$f(x, y) = \frac{x^2 y}{2x^4 + y^2},$$

is shown below. Here it illustrates that the radial limits could conceivably be zero and yet the limit along a parabolic path is not.



The MatLab code used to generate the graph is

```
[x,y] = meshgrid(-3:.08:3);
z=(x.^2.*y)./(2.*x.^4 + y.^2);
surf(x,y,z)
```

2.4 Complex Derivatives

The derivative of a complex-valued function $f(z)$ of a complex variable z is defined as

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

provided that the limit exists. This is just like the definition of the derivative of a real function except that z and h are now complex variables.

Example 2.4.1

Show that $f(z) = \overline{z}$ is not differentiable. To show this, we show that the limit

$$\lim_{h\to 0} \frac{\overline{z+h} - \overline{z}}{h},$$

does not exist. By the properties of conjugation, this simplifies to

$$\lim_{h\to 0} \frac{\overline{h}}{h}.$$

Let $h = x + iy$, so that

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{x+iy\to 0} \frac{x - iy}{x + iy}.$$

Approaching zero along the real axis, we have that $y = 0$, so the limit is

$$\lim_{x\to 0} \frac{x}{x} = 1.$$

Approaching zero along the imaginary axis, we have that $x = 0$, so the limit is

$$\lim_{x\to 0} \frac{-y}{y} = -1.$$

Since the limits do not agree, the limit does not exist, and therefore, $f(z)$ is not differentiable.

It turns out that all of the derivative formulas, such as how to take a derivative of a polynomial, the product rule, the chain rule, and the quotient rule all work the same for complex numbers, provided that the derivative exists in the first place.

2.5 Cauchy-Riemann Equations

Assume the derivative $f'(z)$ of a function $f(z)$ exists. Since $f(z)$ is a complex valued function of a complex variable, it's input is complex and can be written as $z = x + iy$. Likewise, it's value is complex and can be represented by $f(z) = u + iv$. This means u and v are functions of x and y. That is, $u = u(x, y)$ and $v = v(x, y)$, so we can write the function as

$$f(z) = f(x, y) = u(x, y) + i\, v(x, y),$$

where $u(x, y)$ and $v(x, y)$ are real functions.

Then the derivative of $f(z)$ is

$$f'(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h}.$$

Recall that h is a complex variable. We now let $h \to 0$ along two different paths. Approaching zero along the real axis, we have that $h = \alpha + 0 \cdot i$, and our derivative becomes

$$f'(z) = \lim_{\substack{\alpha\to 0\\ \alpha\in\mathbb{R}}} \frac{f(x + \alpha, y) - f(x, y)}{\alpha}.$$

But this is the definition of the partial derivative of $f(z)$ with respect to x. That is,

$$f'(z) = f_x(z) = u_x + iv_x,$$

along this path. Next, we let h approach zero along the imaginary axis. Here, we have that $h = 0 + \beta i$, so our derivative is

$$\begin{aligned} f'(z) &= \lim_{\substack{\beta i\to 0\\ \alpha\in\mathbb{R}}} \frac{f(x, y + \beta) - f(x, y)}{\beta i} \\ &= -i \lim_{\substack{\beta\to 0\\ \alpha\in\mathbb{R}}} \frac{f(x, y + \beta) - f(x, y)}{\beta}. \end{aligned}$$

But this is just $-i$ times the partial derivative of z with respect to y. That is,

$$\begin{aligned} f'(z) &= -i(u_y + iv_y) \\ &= v_y - iu_y. \end{aligned}$$

We have therefore shown that if the derivative of

$$f(z) = f(x, y) = u(x, y) + i\,v(x, y),$$

exists, then necessarily

$$u_x + iv_x = v_y - iu_y.$$

Equating real and imaginary parts gives us the **Cauchy-Riemann equations**

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}$$

otherwise written as

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned}$$

The Cauchy-Riemann equations are necessary conditions for the existence of a unique derivative of $f(z) = u(x, y) + i\,v(x, y)$ at $z = x + iy$. We can show that they are also sufficient conditions if the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous at the point (x, y). We can prove this by looking at the total differential df

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy \\ &= \frac{\partial u}{\partial x}dx + i\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\frac{\partial v}{\partial y}dy \\ &= \frac{\partial u}{\partial x}dx + i\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial x}dy + i\frac{\partial u}{\partial x}dy \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(dx + idy) \\ \frac{df}{dx + idy} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{df}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{df}{dz} &= \frac{\partial f}{\partial x}. \end{aligned}$$

Notice that the righthand side is independent of how dz approaches zero.

The derivative of a complex function may exist for some but not all values of z. In that case, you can use the Cauchy-Riemann equations to find exactly which values of z that the derivative exists.

Example 2.5.1

Show that $f(z) = z^2$ satisfies the Cauchy-Riemann conditions.
Expanding the function, we have that

$$f(z) = z^2 = (x + iy)^2 = x^2 + 2ixy - y^2,$$

Tip

When using the Cauchy-Riemann equations to determine where a complex number is differentiable, be sure to note the domain of the original equation. For example, $f(z) = \ln x + \log y + 2ixy$ is only defined for $x > 0$ and $y > 0$, so that will limit where the function is differentiable.

so $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The partial derivatives are

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \\ \frac{\partial u}{\partial y} &= -2y \\ \frac{\partial v}{\partial x} &= 2y \\ \frac{\partial v}{\partial y} &= 2x \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

the Cauchy-Riemann conditions are satisfied. Since the partial derivatives are continuous, $f(z) = z^2$ is analytic throughout the complex plane, and is therefore, an *entire* function.

Example 2.5.2

Determine if $f(z) = \overline{z} = x - iy$ satisfies the Cauchy-Riemann conditions. Since

$$\begin{aligned} \frac{\partial u}{\partial x} = 1 &\neq \frac{\partial v}{\partial y} - 1 \\ \frac{\partial v}{\partial x} = 0 &= -\frac{\partial u}{\partial y} = 0, \end{aligned}$$

the Cauchy-Riemann conditions are not satisfied.

Polar Form

To derive the Cauchy-Riemann equations in polar form we assume the derivative exists, and we just convert the cartesian form of the Cauchy-Euler equations into polar form. That is, we start with

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}$$

Recall that in Cartesian form, a complex function can be written in the form

$$f(z) = u(x, y) + iv(x, y).$$

In polar form, a function $f(z)$ can be written as

$$f(z) = f\left(re^{i\theta}\right) = U + iV = U(r, \theta) + iV(r, \theta).$$

That is, a complex function in polar form can be written as a real part U plus an imaginry part V, and both U and V are functions of r and θ. In other words,

$$\begin{aligned} u(x, y) &= U(r, \theta) = U(r(x, y), \theta(x, y)) \\ v(x, y) &= V(r, \theta) = V(r(x, y), \theta(x, y)). \end{aligned}$$

Differentiating this with the chain rule gives us

$$\begin{aligned} u_x &= U_r r_x + U_\theta \theta_x \\ u_y &= U_r r_y + U_\theta \theta_y \\ v_x &= V_r r_x + V_\theta \theta_x \\ v_y &= V_r r_y + V_\theta \theta_y. \end{aligned}$$

Now we just have to find all these partial derivatives.

Recall the transformation equations

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x}, \end{aligned}$$

that is, r and θ are functions of x and y. Computing all the relavant partial derivatives, we get

$$\begin{aligned} r_x &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \\ r_y &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta \\ \theta_x &= -\frac{y}{x^2+y^2} = -\frac{1}{r}\sin\theta \\ \theta_y &= \frac{1}{r}\cos\theta. \end{aligned}$$

Plugging all these in gives us

$$\begin{aligned} u_x &= U_r\cos\theta - U_\theta\frac{1}{r}\sin\theta \\ u_y &= U_r\sin\theta + U_\theta\frac{1}{r}\cos\theta \\ v_x &= V_r\cos\theta - V_\theta\frac{1}{r}\sin\theta \\ v_y &= V_r\sin\theta + V_\theta\frac{1}{r}\cos\theta. \end{aligned}$$

Now, we just plug these results into the Cartesian form of the Cauchy-Riemann equations

$$\begin{aligned} U_r\cos\theta - U_\theta\frac{1}{r}\sin\theta &= V_r\sin\theta + V_\theta\frac{1}{r}\cos\theta \\ V_r\cos\theta - V_\theta\frac{1}{r}\sin\theta &= -\left(U_r\sin\theta + U_\theta\frac{1}{r}\cos\theta\right). \end{aligned}$$

Now we multiply the top equation by $\cos\theta$ and the bottom equation by $\sin\theta$ and add the two and simplify to get

$$U_r = \frac{1}{r}V_\theta.$$

Then we multiply the top equation by $\sin\theta$ and the bottom equation by $\cos\theta$ and subtract the bottom one from the top one to get

$$\frac{1}{r}U_\theta = -V_r.$$

Doing some rearranging of these two equations gives us the Cauchy-Riemann equations in polar form

$$\begin{aligned} rU_r &= V_\theta \\ U_\theta &= -rV_r. \end{aligned}$$

To remember these, note that the r multiplies the two partial derivatives of r.

In summary, if all of the partial derivatives exist, then the statement $f'(z)$ exists, is equivalent to the Cauchy-Riemann equations in Cartesian and polar form. To differentiate a function where the derivative is known to exist, we can just use the Calculus I rules that we use for real functions. If we cannot differentiate it using those rules, we can also use the following equations to compute the derivatives

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= e^{-i\theta}\left(U_r + iV_r\right). \end{aligned}$$

2.6 Harmonic Functions

Common notation for **Laplace's equation** in two dimensions is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \nabla^2 u &= 0 \\ \Delta u &= 0. \end{aligned}$$

where $u = u(x, y)$.

A function $u(x, y)$ is a **harmonic function** on $U \subset \mathbb{C}$ if it satisfies Laplace's equation at every point (x, y) in U.

The real and imaginary parts of an analytic function separately satisfy Laplace's equation. That is, if $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function then u and v are both harmonic functions,

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0. \end{aligned}$$

The first can be proven by partially differentiating both Cauchy-Riemann conditions with respect to x and equating them. Similarly, the second can be proven by partially differentiating them with respect to y and equating them. If this is the case, then u and v are called **harmonic conjugates** of each other.

Example 2.6.1

Show that $u(x, y) = e^x \sin y$ is a harmonic function and then find $v(x, y)$ such that $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function.

To be a harmonic function, u must satisfy Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

Taking the partial derivatives, we get

$$\begin{aligned} u_x &= e^x \sin y \\ u_{xx} &= e^x \sin y \\ u_y &= e^x \cos y \\ u_{yy} &= -e^x \sin y. \end{aligned}$$

We see that u satisfies Laplace's equation, so it is a harmonic function. To find v such that f is an analytic function, we use the Cauchy-Riemann conditions, which tell us that

$$u_x = e^x \sin y = v_y.$$

Since $v_y = e^x \sin y$, we can partially integrate $e^x \sin y$ with respect to y to find v. Doing so gives us

$$v(x,y) = \int e^x \sin y \, dy = -e^x \cos y + C(x).$$

Notice that instead of a constant C, our unknown could be a function of the variable x, which we had held constant.

The second Cauchy-Riemann condition tells us that

$$u_y = e^x \cos y = -v_x.$$

Since $v_x = -e^x \cos y$, we can partially integrate $-e^x \cos y$ with respect to x to find v. Doing so gives us

$$v(x,y) = \int -e^x \cos y \, dx = -e^x \cos y + C(y).$$

Comparing our two results for $v(x,y)$, we see that $C(x) = C(y) = 0$, and so

$$v(x,y) = -e^x \cos y,$$

and our analytic function is

$$f(z) = e^x \sin y - i\, e^x \cos y.$$

2.7 Analytic functions

A complex valued function f of a complex variable is **analytic** at z_0 if there exists a radius $\varepsilon > 0$ such that the derivative $f'(z)$ exists for all $z \in N_\varepsilon(z_0)$. In other words, $f(z)$ is analytic at a point z_0 if its derivative exists for every point in some neighborhood of z_0. This implies that if $f(z)$ is analytic at z_0, then $f(z)$ is analytic at every z in a disk neighborhood of z_0.

The difference between being analytic at a point and being differentiable at a point, is that to be analytic at a point, a function must be differentiable at every point in some disk neighborhood of the point. A function cannot be analytic at a single point, but it can be differentiable at a single point. Since there are uncountably infinite points in any disk neighborhood, this implies that if a function is differentiable only at a countably infinite number of points, then it is analytic nowhere. For example, to show that a function is nowhere analytic, we have to show that the function is not differentiable at all points in any arbitrary disk. So a function that is differentiable at finite or *countably* infinite points is analytic nowhere.

If a function is analytic at every point in the neighborhood of z_0 except exactly z_0, then z_0 is a **singular point** or a **singularity** of the function. For example, $f(z) = \frac{1}{z}$ is analytic everywhere except at $z = 0$, which is a singularity. A singularity is basically a point at which the function is not well-behaved. For example, if the function is undefined or non-differentiable at that point, the point is called a singularity. In contrast, a **regular** point is a point where $f(z)$ is analytic.

If a function $f(z)$ is single-valued and differentiable at every point of a domain D, except for a finite number of singular points, the function is considered **analytic** in D. If there are no singular points in D, the function is said to be **regular** or **holomorphic** in D.

If a function $f(z)$ is analytic in $N^*_\varepsilon(z_0)$ but $f'(z_0)$ does not exist, then z_0 is an **isolated**

singularity. In other words, if a point z_0 is the only place in its disk neighborhood where $f(z)$ is not analytic, then z_0 is an isolated singularity.

If $f(z)$ is analytic at every point $D \subset \mathbb{C}$, then $\exists$ an open set $U \supset D$ such that $f'(z)$ exists $\forall z \in U$. In other words, if $f(z)$ is analytic in some region D of the complex plane, then there exists a larger region U containing D such that the derivative of $f(z)$ exists at every point in U. For example, consider a single line D along which $f(z)$ is analytic. Then by definition of analyticity, $f'(z)$ exists within a small disk around every point along D. Therefore, there is a larger region completely containing D in which $f'(z)$ exists.

If $f : U \to \mathbb{C}$ is analytic on the connected region U, and if $f'(z) = 0$ for all z in U, then $f(z) =$ constant on all of U.

If $g : U \to \mathbb{C}$ is analytic in U and $z_n, z_0 \in U$ with $\lim_{n\to\infty} z_n = z_0$, and if $f(z_n) = g(z_n)$ for $n = 1, 2, 3, \ldots$, then $f(z) = g(z)$ for all $z \in U$. What this tells us is that if there is an infinite sequence of points z_n approaching some fixed point z_0 in the complex plane, and if for every z_n, the two functions f and g evaluated at z_n give the same results, then f and g are really the same function.

This is not the case for real functions. Consider the functions $f(x) = 0$ and $g(x) = x^2 \sin \frac{\pi}{x}$. Now consider the points $x_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ If we evaluate f and g at each of these points the results are always the same—0. However, f and g are not at all the same function.

A corrollary of this fact is that if f and g are analytic on U where $[a, b] \subset U$ with $f(x) = g(x)$ for all $a \le x \le b$, then $f(z) = g(z)$ for all $z \in U$. In other words, if U is a region of the complex plane containing at least the portion of the real number line $[a, b]$, and f and g are analytic and $f(x) = g(x)$ for all $x \in [a, b]$, then f and g are the same functions over the entire region U. This property gives us an easy way to extend properties of real functions to their complex versions. Because of this property, all the trig identities that we are used to also hold for complex values.

Example 2.7.1

Verify that $\cos^2 z + \sin^2 z = 1$ for all complex numbers.

We start by noting that $\cos z$ and $\sin z$ are analytic in the whole complex plane (which naturally includes all the real numbers in any interval $[a, b]$. Next, we define

$$\begin{aligned} f(z) &= \cos^2 z + \sin^2 z - 1 \\ g(z) &= 0. \end{aligned}$$

We want to show that $f(z) = g(z)$ on some interval in $\mathbb{R}$, which means we want to show that $f(x) = g(x)$ for some interval in x. But from $\sin^2 x + \cos^2 x = 1$ we know that $f(x) = 0$ for all real numbers, therefore, $f(x) = g(x)$ for all real numbers. Therefore, from the corollary discussed above, we know that $f(z) = g(z)$ on the entire complex plane and therefore

$$\cos^2 z + \sin^2 z = 1.$$

An **entire function** is one that is analytic in the whole complex plane.

If two functions are analytic, then their compositions are analytic. For example,

$$f(z) = e^{\frac{1}{z}},$$

is analytic except at $z = 0$ because e^z is analytic everywhere and $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

2.8 Summary: Analytic Functions

Regions of the Complex Plane

A **closed set** is one that contains its boundary points. An **open set** does not. A set is **connected** if for any pair of points in the set, there is a polygonal path *entirely within* the set that goes from one point to the other. A set is **bounded** if it lies completely within some circle. A **domain** is a set that is open and connected.

Mappings

To find the **image** of a region, first note the exact x and y values of the boundary of the region. Then determine what the complex function, which maps the region to the image, does to each boundary.

For the squaring or the principal square root function, use the polar form. For example, the squaring function $f(z) = z^2$ can be written as

$$f(re^{i\theta}) = r^2 e^{2i\theta}.$$

From this, it is apparent that whatever the magnitude and angle of the input is, the output magnitude is squared and the angle is doubled.

For the exponential function, write the domain in Cartesian form and the image in polar form as

$$f(x + iy) = e^{x+iy} = e^x e^{iy}.$$

Notice that e^x gives the magnitude r of the image and that y gives the angle θ of the image. If it helps, remember that e^x is entirely real, and e^{iy} traces out a circle in the complex plane as y is varied. So if y is fixed and x is varied, then the image is a straight line extending away from the origin. If x is fixed and y is varied, then the image is a circle centered at the origin with radius e^x.

Limits

For complex limits, there are infinite directions to approach a complex number z_0. For the limit to exist, the limit as you approach z_0 along every possible path, must be the same. If any two curves do not agree on the limit, then the limit does not exist.

In other words, we cannot easily prove that a limit exists, but we only have to obtain two different limits to prove that it does not exist. One way to do that is to take several **radial limits**. A radial limit is just the limit as you approach zero along some straight line.

To compute a limit, begin by replacing z with $x+iy$. Then the radial limit along the real axis, means $y = 0$. Similarly, the radial limit along the imaginary axis means $x = 0$. The radial limit along the line $y = x$ means $y = x$. In all three of these cases, the limit simplifies to a limit in one variable. The limit along any path can be obtained using the equation of the path. For example, to calculate the limit approaching zero along the path $y = x^2$, just replace all y's in the limit by x^2.

Derivatives

The derivative of a complex-valued function $f(z)$ of a complex variable z is defined as

$$f'(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h},$$

provided that the limit exists. This is just like the definition of the derivative of a real function except that z and h are now complex variables.

Cauchy-Riemann Equations

If and only if the Cauchy-Riemann equations are satisfied at the points z_k, then the derivative of a complex function $f(z)$ exists at those points.

To use the Cauchy-Riemann equations, we first write the function in terms of a real part and an imaginary part

$$f(z) = u(x, y) + iv(x, y).$$

To do that, let $z = x + iy$ then separate the real and imaginary parts. Then the Cauchy-Riemann equations are

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

In polar form, we write the function in terms of a real part and an imaginary part as

$$f(z) = U(r, \theta) + iV(r, \theta).$$

To do that, let $z = re^{i\theta}$, then the Cauchy-Riemann equations are

$$\begin{aligned} rU_r &= V_\theta \\ U_\theta &= -rV_r. \end{aligned}$$

If a function satisfies the Cauchy-Riemann conditions, then we are allowed to differentiate it. To differentiate a complex function, we just use the familiar rules for differentiation. If we cannot differentiate it using those rules, we can also use the following equations to compute the derivatives

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= e^{-i\theta}\left(U_r + iV_r\right). \end{aligned}$$

Harmonic Functions

A function $u(x, y)$ is a **harmonic function** on $U \subset \mathbb{C}$ if it satisfies Laplace's equation at every point (x, y) in U. That is,

$$u_{xx} + u_{yy} = 0.$$

To check if a function is harmonic, just take the second derivatives with respect to x and y and equate them to see if they sum to zero.

If $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function then u and v are both harmonic functions. That is,

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0. \end{aligned}$$

Then u and v are called **harmonic conjugates**.

If you're given u, you can find its harmonic conjugate v by using the Cauchy-Riemann equations. Since you're given u, you can calculate u_x and u_y, then by the Cauchy-Riemann conditions,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Partially integrate the first equation with respect to y then you'll have v but with some unknown function $C(x)$. Next, partially integrate the second equation with respect to x then you'll have v but with some unknown function $C(y)$. Then you can compare the two results to deduce the values of $C(x)$ and $C(y)$.

Analytic Functions

A function $f(z)$ is analytic at a point z_0 if its derivative exists for every point in some neighborhood of z_0.

To determine where a complex function is differentiable, use the Cauchy-Riemann conditions. Then the function is analytic in every *open* region within which it is differentiable. If it is only differentiable at a finite or countably infinite number of points, then it is nowhere analytic.

Chapter 3

Elementary Functions

3.1 The Exponential Function

Properties

The complex exponential function is defined as

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Its properties include (z and w are complex numbers and θ is a real number)

$$\begin{aligned}
e^{z+w} &= e^z e^w \\
e^{z-w} &= \frac{e^z}{e^w} \\
e^{-z} &= \frac{1}{e^z} \\
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{iz} &= \cos z + i\sin z \\
e^z &= e^{z+2\pi ik}, \qquad k \in \mathbb{Z}.
\end{aligned}$$

The last property tells us that the exponential function is periodic with the imaginary period $2\pi i$.

Notice that $(e^z)^w = e^{zw}$ is not included as a property of the exponential function. In fact, in general, it is not true. So far we have not defined raising complex numbers to complex powers, but beyond that, $(e^z)^w$ is a multi-valued function.

Derivative

We can use the Cauchy-Riemann equations to show that the exponential function is differentiable. We first write it as the sum of real and imaginary parts as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos\theta + i\sin\theta),$$

then

$$\begin{aligned}
u(x,y) &= e^x\cos\theta \\
v(x,y) &= e^x\sin\theta.
\end{aligned}$$

Taking the appropriate partial derivatives, we find that e^z is differentiable everywhere. Since it is analytic on all $\mathbb{C}$, it is an entire function.

Its derivative is

$$\begin{aligned}\frac{d}{dz}e^z &= u_x + iv_x \\ &= e^x \cos y + ie^x \sin y \\ &= e^z.\end{aligned}$$

Example 3.1.1

Show that

$$\left|e^{z^2}\right| \leq e^{|z|^2}.$$

When doing this kind of thing, we convert the complex variable z to real variables via $z = x + iy$ so that we can more easily deduce what each side of the inequality means. The left side is

$$\begin{aligned}\left|e^{z^2}\right| &= \left|e^{(x+iy)^2}\right| \\ &= \left|e^{x^2-y^2+2xyi}\right| \\ &= \left|e^{x^2-y^2}e^{2xyi}\right| \\ &= \left|e^{x^2-y^2}\right|\left|e^{2xyi}\right| \\ &= e^{x^2}e^{-y^2}.\end{aligned}$$

Here we made use of the fact that $2xyi$ is some imaginary number, and so e^{2xyi} lies on the unit circle. Therefore, $\left|e^{2xyi}\right| = 1$. Also, we know that $\left|e^{x^2-y^2}\right| = \left|e^{x^2}\right|\left|e^{-y^2}\right|$, but since the exponential function is always positive, this is just $e^{x^2}e^{-y^2}$.

We do the right side in a similar manner to get

$$\begin{aligned}e^{|z|^2} &= e^{|x+iy|^2} \\ &= e^{x^2+y^2} \\ &= e^{x^2}e^{y^2}.\end{aligned}$$

By the properties of the real exponential function, we know that

$$e^{-y^2} \leq e^{y^2}.$$

After multiplying both sides by e^{x^2}, the inequality given at the beginning follows.

Mapping

To examine the mapping properties of the exponential function $w = f(z) = e^z$, we start by writing the domain in cartesian coordinates,

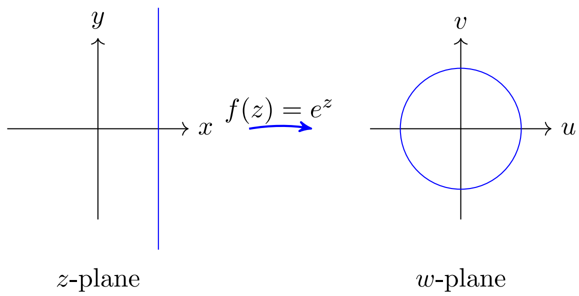
$$z = x + iy,$$

and the image in polar coordinates,

$$f(z) = e^{x+iy} = e^x e^{iy}.$$

Notice that e^x gives the magnitude r of the image and that y gives the angle θ of the image. If it helps, remember that e^x is entirely real, and e^{iy} traces out a circle in the complex plane.

If we fix x and vary y, then in the z-plane, we have a vertical line. To understand how the image is formed, we note that $r = e^x$ remains fixed since x is fixed and e^{iy} varies since y varies. So the image of a vertical line is a circle.

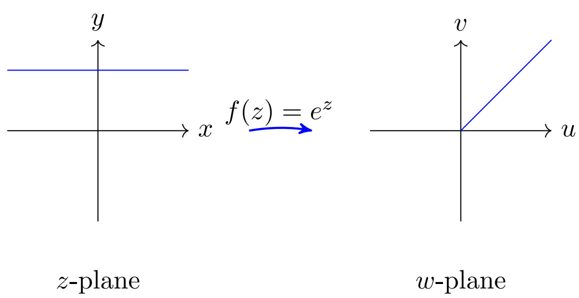


Note that as y goes from $y = 0$ to $y = 2\pi i$, the image in the w plane is the complete circle. As y goes above $2\pi i$. The circle is traced out again. We say that e^z is periodic with period $2\pi i$. That is,

$$e^z = e^{z+2\pi i}.$$

If the vertical line in the z-plane is at $x = 1$, then the radius of the circle in the w-plane is e^x. As x is increased, the radius of the circle is increased. At $x = 0$, the radius of the circle is $r = e^0 = 1$. For vertical lines to the left of $x = 0$, the radii become smaller and smaller, tending toward zero.

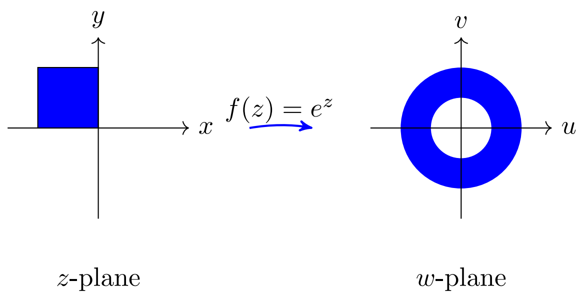
If we fix y and allow x to vary, then on the z-plane, we have a horizontal line. On the w-plane, the distance from the origin varies as $r = e^y$ while the angle is fixed. As $x \to \infty$, $e^x \to \infty$, and as $x \to -\infty$, $e^x \to 0$. So on the w-plane we have a ray pointing at an angle y.



Notice that when $y = 0, 2\pi, 4\pi, \ldots$ the image is the real numbers, that is, the ray extending from $(0, 0)$ with angle 0. Also, when $y = \pi, 3\pi, 5\pi, \ldots$ the image is the imaginary numbers, that is, the ray extending from $(0, 0)$ with angle π. Between the horizontal lines $y = 0$ and $y = 2\pi$ on the z-plane, all the complex numbers are covered in the w-plane. Between the horiztonal lines $y = 2\pi$ and $y = 4\pi$ on the z-plane, all the complex numbers are covered again in the w-plane. This is again a manifestation of the periodicity of e^x.

A rectangle in the z-plane maps to a circle or a portion of a circle. For example, if given a rectangle on the z-plane with dimensions $-1 \leq x \leq 0$ and $0 \leq y \leq 2\pi$, then you know that the distance from the origin will be $e^{-1} \leq r \leq e^0$ and the angle will be $0 \leq \theta \leq 2\pi$, so the image of this rectangle is the annulus with outer radius 1 and inner radius $\frac{1}{e}$.

$$\begin{aligned} -1 \leq x \leq 0 \quad &\Longleftrightarrow \quad e^{-1} \leq r \leq e^0 \\ 0 \leq y \leq 2\pi \quad &\Longleftrightarrow \quad 0 \leq \theta \leq 2\pi. \end{aligned}$$



A non-horizontal line in the z-plane maps to a spiral in the w-plane.

3.2 Logarithmic Functions

The complex natural logarithm is the function $\log z = w$ with the property that $z = e^w$, where z and w are complex numbers. Note that if $z = e^w$ then $z = e^{w+2\pi in}$ for all $n \in \mathbb{Z}$. That is, there are infinitely many w that satisfy $z = e^w$. So the logarithm of a complex number is a **multi-valued function**.

For example, to find $\log(-2) = w$, we want to find the number w such that $e^w = -2$. If $w = a + bi$, then $e^w = e^a e^{bi} = -2$ if $e^a = 2$ and $e^{bi} = -1$. In other words, $a = \ln 2$ and $b = \pi$. So we have that

$$\log(-2) = \ln 2 + \pi i.$$

However, adding any integer multiple of $2\pi i$ to w in e^w gives the same number, so in fact,

$$\log(-2) = \ln 2 + \pi i + 2\pi in, \qquad n \in \mathbb{Z}.$$

Since $b = \pi$ is the principal argument of $-2 = e^w$, we call $w = \ln 2 + \pi i$ the principal logarithm of -2. Other values such as $w = \ln 2 + 3\pi i$ and $w = \ln 2 - \pi i$ are non-principal logarithms of -2.

If $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$ (i.e. θ is the principal argument of z), or if $z = x + iy$, then the **principal natural logarithm** of z can be defined in any of the following ways

$$\begin{aligned} \operatorname{Log} z &= \ln|z| + i \operatorname{Arg} z \\ &= \ln r + i\theta \\ &= \frac{1}{2}\ln(x^2 + y^2) + i \operatorname{Arctan}\left(\frac{y}{x}\right). \end{aligned}$$

Notice that we denote the principal logarithm with an uppercase 'L'. Non-principal logarithms are denoted $\log z$.

To confirm that our formula for the logarithm is correct, we note that

$$e^{\operatorname{Log} z} = e^{\ln r + i\theta} = e^{\ln r} e^{i\theta} = re^{i\theta} = z.$$

Derivative

To see if the complex logarithm is differentiable, we use the polar form of the Cauchy-Riemann equations. If $\operatorname{Log} z = \ln r + i\theta$, then $U(r, \theta) = \ln r$ and $V(r, \theta) = \theta$. Applying the Cauchy-Riemann conditions demonstrates that $\operatorname{Log} z$ is differentiable everywhere. In fact, $\operatorname{Log} z$ is analytic everywhere except at 0 and the slit along the negative real numbers. The derivative exists along the negative real numbers, but there is a jump there.

The derivative of $\operatorname{Log} z$ is

$$\begin{aligned}\frac{d}{dz}\operatorname{Log} z &= e^{-i\theta}\left(U_r + iV_r\right) \\ &= e^{-i\theta}\left(\frac{1}{r} + i\cdot 0\right) \\ &= \frac{1}{re^{i\theta}} \\ &= \frac{1}{z}.\end{aligned}$$

Non-principal Branches

Recall that $\operatorname{Log} z$ is analytic everywhere except on the ray pointing along the negative real axis. We can define non-principal branches by

1. choosing a ray along which to slit $\mathbb{C}$, and
2. identifying the angle α from the positive real axis to the slit.

Then this branch of the logarithm is a single-valued function defined as

$$\log z = \ln|z| + i\theta,$$

with θ between α and $\alpha + 2\pi$. We know that it *is* a logarithm because it satisfies the required property that $e^{\log z} = z$.

Properties

Recall the properties of logarithms of a real variable

$$\begin{aligned}\log xy &= \log x + \log y \\ \log\frac{x}{y} &= \log x + \log y \\ \log x^n &= n\log x.\end{aligned}$$

In general, these properties do not hold for complex variables. The results are similar, but they often differ by a term such as $2\pi ik$ where $k \in \mathbb{Z}$. This means we have to be careful when trying to use log properties when it is of a complex variable. The safest way is simply to use the formula $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ and simplify.

3.3 Power Functions

Recall that for a real variable $x > 0$, we can define x raised to a constant real number c as $x^c = e^{c\ln x}$. We use the same definition to define what it means to raise a complex variable z to a complex number C

$$z^C = e^{C\log z}.$$

In particular, the principal value of z^C is denoted

$$\text{PV-}\ z^C = e^{C\operatorname{Log} z}.$$

The derivative formula that we are used to still works. That is,

$$\frac{d}{dz}z^C = Cz^{C-1}.$$

In general, the properties $(xy)^c = x^c y^c$ and $(x^c)^d = x^{cd}$ do not hold for complex numbers.

The derivative of a complex constant raised to a complex variable is calculated in the same way as with real variables

$$\frac{d}{dz}c^z = c^z \log z,$$

however, it might be off by $2\pi ik$.

Example 3.3.1

Find the principal value of i^i.
We know that

$$\text{PV- } z^C = e^{C \operatorname{Log} z}.$$

In this case, $z = C = i$ and $\operatorname{Log} i = \ln|i| + i\theta = \ln 1 + \frac{\pi}{2}i = \frac{\pi}{2}i$, so we get

$$\text{PV- } i^i = e^{i\frac{\pi}{2}i} = e^{-\frac{\pi}{2}}.$$

More generally,

$$i^i = e^{-\left(2\pi k + \frac{\pi}{2}\right)}.$$

3.4 Trig Functions

The complex sine and cosine functions are defined in terms of series as

$$\begin{aligned} \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}. \end{aligned}$$

From this, we can easily show that

$$\begin{aligned} \sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\ \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right). \end{aligned}$$

The derivatives of the complex trig functions work just like the derivatives of the real trig functions. Also, the complex trig functions are periodic in the same way that their real counterparts are.

Example 3.4.1

What are the real and imaginary components of $\sin(a + bi)$?
Using the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

we can rewrite this as

$$\sin(a + bi) = \sin a \cos(bi) + \cos a \sin(bi).$$

By plugging ib and $-ib$ into Euler's formula, we get

$$\begin{aligned} e^{-b} &= \cos(ib) + i\sin(ib) \\ e^{b} &= \cos(ib) - i\sin(ib). \end{aligned}$$

Adding the two together gives

$$\begin{aligned} e^b + e^{-b} &= 2\cos(ib) \\ \cos(ib) &= \frac{1}{2}\left(e^b + e^{-b}\right) \\ \cos(ib) &= \cosh b. \end{aligned}$$

Subtracting the first from the second gives

$$\begin{aligned} e^b - e^{-b} &= -2i\sin(ib) \\ \sin(ib) &= i\frac{1}{2}\left(e^b - e^{-b}\right) \\ \sin(ib) &= i\sinh b. \end{aligned}$$

Plugging these in gives us

$$\sin(a+bi) = \sin a \cosh b + i\cos a \sinh b,$$

so

$$\begin{aligned} \Re\left[\sin(a+bi)\right] &= \sin a \cosh b \\ \Im\left[\sin(a+bi)\right] &= \cos a \sinh b. \end{aligned}$$

3.5 Hyperbolic Trig Functions

The complex hyperbolic functions are defined as

$$\begin{aligned} \sinh z &= \frac{1}{2}\left(e^z - e^{-z}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\ \cosh z &= \frac{1}{2}\left(e^z + e^{-z}\right) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \end{aligned}$$

It is easy to show that

$$\begin{aligned} \cos(iz) &= \cosh z \\ \cosh(iz) &= \cos z. \end{aligned}$$

This allows us to take the cosine of imaginary angles.

One key difference between the complex hyperbolic functions and the real hyperbolic functions is that the complex ones are periodic with period $2\pi i$. That is,

$$\cosh(z + 2\pi i) = \cosh z.$$

3.6 Summary: Elementary Functions

Exponential Functions

Properties of e^z include (z and w are complex numbers and θ is a real number)

$$\begin{aligned} e^{z+w} &= e^z e^w \\ e^{z-w} &= \frac{e^z}{e^w} \\ e^{-z} &= \frac{1}{e^z} \\ e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{iz} &= \cos z + i\sin z \\ e^z &= e^{z+2\pi i k}, \qquad k \in \mathbb{Z}. \end{aligned}$$

The last property tells us that the exponential function is periodic with the imaginary period $2\pi i$.

It is analytic on all $\mathbb{C}$ and therefore an entire function. Its derivative is

$$\frac{d}{dz}e^z = e^z.$$

When verifying absolute value inequalities involving exponential functions, let $z = x + iy$ and you can often break it into parts and use the fact that the modulus of e raised to any *imaginary* number is just 1.

Logarithmic Functions

If $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$ (i.e. θ is the principal argument of z), or if $z = x + iy$, then the **principal natural logarithm** of z can be defined in any of the following ways

$$\begin{aligned} \operatorname{Log} z &= \ln|z| + i \operatorname{Arg} z \\ &= \ln r + i\theta \\ &= \frac{1}{2}\ln(x^2+y^2) + i \operatorname{Arctan}\left(\frac{y}{x}\right). \end{aligned}$$

$\operatorname{Log} z$ is differentiable everywhere. Furthermore, $\operatorname{Log} z$ is analytic everywhere except at 0 and the slit along the negative real numbers. The derivative exists along the negative real numbers, but there is a jump there. The derivative is

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

We can define non-principal branches of the logarithm by choosing a ray along which to slit $\mathbb{C}$ and identifying the angle α from the positive real axis to the slit. Then this branch of the logarithm is a single-valued function defined as $\log z = \ln|z| + i\theta$, with θ between α and $\alpha + 2\pi$.

In general, the properties of logarithms that we are used to with logarithms of real numbers do not hold for complex numbers. The safest way to deal with complex logarithms is simply to use the formula $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ and simplify.

Power Functions

The principal value of a complex variable z raised to a complex number C is defined as

$$\text{PV-}\ z^C = e^{C \operatorname{Log} z} = \exp\left(C(\ln|z| + i \operatorname{Arg} z)\right).$$

The derivative formula that we are used to still works. That is,

$$\frac{d}{dz}z^C = Cz^{C-1}.$$

Trig and Hyperbolic Trig Functions

$$\begin{aligned} \sin z &= \frac{e^{iz}-e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz}+e^{-iz}}{2} \\ \sinh z &= \frac{e^z-e^{-z}}{2} \\ \cosh z &= \frac{e^z+e^{-z}}{2} \\ \cos^2 z + \sin^2 z &= 1 \\ \cosh^2 z - \sinh^2 z &= 1. \end{aligned}$$

The derivatives of the complex trig functions work just like the derivatives of the real trig functions. Also, the complex trig functions are periodic in the same way that their real counterparts are. One key difference between the complex hyperbolic functions and the real hyperbolic functions is that the complex ones are periodic with period $2\pi i$. For example, $\cosh(z+2\pi i) = \cosh z$.

By replacing z with iz in the exponential definitions given above, it is easy to show that

$$\begin{aligned} \sin(iz) &= i\sinh z \\ \cos(iz) &= \cosh z \\ \sinh(iz) &= i\sin z \\ \cosh(iz) &= \cos z. \end{aligned}$$

By using the double angle formulas

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B, \end{aligned}$$

we can write $\sin z$ and $\cos z$ in real and imaginary parts as

$$\begin{aligned}\sin(x+iy) &= \sin x\cosh y + i\cos x\sinh y\\ \cos(x+iy) &= \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

Chapter 4

Integration

4.1 Integrals of a Real Variable

Integration of a real-valued function of a real variable is an integral over the real number line. That is, the domain of the functions are only along the real number line. For complex valued functions of a complex variable, the domain is two-dimensional, so an integral of such a function is a line integral over a path (or contour) in the complex plane.

Recall that the derivative of a complex-valued function of a real variable, for example, $g(t) = e^{it}$ can be found in two different ways. We can write g in terms of u and v as $g(t) = \cos t + i \sin t$, then the derivative of g is just the derivative of u plus i times the derivative of v. That is, if $g(t) = u(t) + iv(t)$, then

$$\frac{d}{dt}g(t) = \frac{d}{dt}u(t) + i\frac{d}{dt}v(t).$$

In our case, $g'(t) = -\sin t + i\cos t = -\cos t - i\sin t = i(\cos t + i\sin t) = ie^{it}$. However, we can also directly differentiate this kind of function as $g'(t) = ie^{it}$ by using the chain rule. Notice that $g(t)$ defines a contour in $\mathbb{C}$. As t goes from 0 to π, for example, $g(t)$ goes along the upperhalf of the unit circle in $\mathbb{C}$. We have, in effect, taken the derivative of a contour.

The integral of a complex-valued function of a real variable can also be performed by breaking the function into real and complex components and doing each component individually. That is, if $g(t) = u(t) + iv(t)$ is a contour in $\mathbb{C}$ with the endpoints a and b on the real axis, then

$$\begin{aligned}\int_a^b g(t)\,dt &= \int_a^b [u(t) + iv(t)]\,dt \\ &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.\end{aligned}$$

Example 4.1.1

Calculate the integral of e^{it} from 0 to π.

Recall that that e^{it} is the unit circle in the complex plane. So from $t = 0$ to $t = \pi$, the function e^{it} traces out the upper half of the unit circle.

$$\int_0^\pi e^{it}\,dt = \int_0^\pi \cos t\,dt + i\int_0^\pi \sin t\,dt = 2i.$$

Example 4.1.2

Calculate the integral

$$\int_0^\infty e^{-zt}\,dt.$$

We can compute such an improper integral in much the same way as we did in calculus, but we have to be careful with the limit at infinity when dealing with complex numbers. Integrating, we have that

$$\int_0^\infty e^{-zt}\,dt = -\frac{1}{z}e^{-zt}\Big|_0^\infty = -\frac{1}{z}\lim_{z\to\infty} e^{-zt} + \frac{1}{z}.$$

Note that we can expand e^{-zt} as $e^{-xt}(\cos t + i\sin t)$. Now as $t\to\infty$, the sin and cosine terms oscillate between 1 and -1, but e^{-xt} quickly approaches 0 *provided that* $x \geq 0$, so overall e^{-zt} approaches 0 provided that the same condition holds. So our integral is

$$\int_0^\infty e^{-zt}\,dt = \frac{1}{z}, \qquad \text{for Re } z \geq 0.$$

4.2 Contour Integrals

We are primarily interested in the integration of a complex-valued function of a complex variable. Since the variable is complex (i.e. 2-dimensional) the integration is over a path in the complex plane rather than over the line of real numbers.

A **contour integral** is an integral of a complex function over a path C in the complex plane. It can be defined as the limit of the Riemann partial sum

$$S_n = \sum_{j=1}^{n} f(\zeta_i)\Delta z_j,$$

where $\Delta z_j = z_j - z_{j-1}$, and ζ_i is a point on the contour C between z_j and z_{j-1}. Then

$$\int_C f(z)\,dz = \lim_{n\to\infty}\lim_{\Delta z_j\to 0}\sum_{j=1}^{n} f(\zeta_i)\Delta z_j.$$

The contour integral is always the Riemann sum, however, there are easier ways to compute contour integrals.

If we parametrize the path C as a function of t with endpoints a and b, then

$$\int_C f(z)\,dz = \int_a^b f(C(t))C'(t)\,dt.$$

Notice that we are making the substitution $z = C(t)$, then $dz = C'(t)\,dt$. Notice that the integrand on the right is just a function of t, that is $f(C(t))C'(t) = g(t)$ is a complex valued function of a real variable. But we can also split such a function up as a real part and an imaginary part, and so

$$\int_a^b g(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

In general, when integrating a complex function over a circular path, we can parametrize a circle of radius r centered at z_0 as

$$z = C(t) = z_0 + re^{it}.$$

Note, this also works if the path is only a portion of a circle because we can choose the t values to run through. If integrating a function over a line segment running from a point z_0 to a point z_1, we can parametrize the line segment as

$$z = C(t) = (1-t)z_0 + tz_1.$$

If the path is composed of multiple straight line segments, we just break the integral into as many parts as there are line segments, compute each integral separately using the parametrization for that specific line segment, then add the results together to get the value of the integral over all the line segments.

Example 4.2.1

Compute

$$\int_C \frac{z+2}{z} dz,$$

where C is the upper semicircle of radius 2 traveled in the counterclockwise direction.

We can parametrize this semicircle as $C(t) = 2e^{it}$ where t goes from 0 to π. We make the substitutions $z = C(t) = 2e^{it}$ and $dz = C'(t) = 2ie^{it}\,dt$, then our integral becomes

$$\begin{aligned} \int_0^\pi \frac{2e^{it}+2}{2e^{it}} 2ie^{it}\,dt &= 2i \int_0^\pi \left(e^{it}+1\right) dt \\ &= -4 + 2\pi i. \end{aligned}$$

The direction of a contour matters. The integral over some contour C is the negative of the integral over the same contour C taken in the opposite direction. We denote this by

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz.$$

We can also add contours together to form a longer contour. For example, if C_+ is an upper semicircle of some radius, C_- is the lower semicircle of the same radius, and C is the circle of the same radius, all of which are centered at the same place in $\mathbb{C}$, then

$$\int_C f(z)\,dz = \int_{C_+} f(z)\,dz + \int_{C_-} f(z)\,dz.$$

Note, the direction of each contour must be the same—all counterclockwise or all clockwise.

Example 4.2.2

If the value of

$$\int_C f(z)\,dz,$$

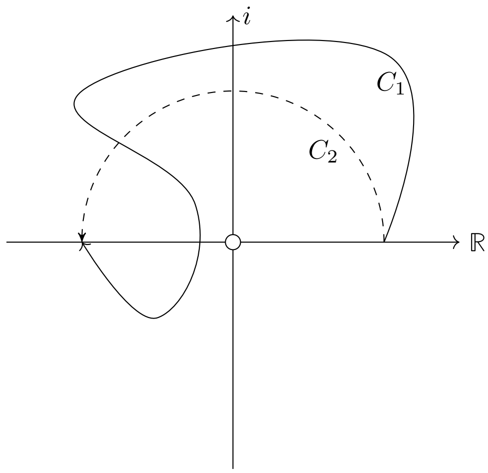
is $-4+2\pi i$ when C is the upper semicircle of radius 2 taken in the counterclockwise direction and its value is $-4-2\pi i$ when C is the lower semicircle of the same radius taken in the *clockwise* direction, what is the value of the integral if C is the circle of radius 2 taken in the counterclockwise direction?

The integral over the circle is the sum of the integrals over the upper and lower semicircles, but the answer given for the lower one is for the clockwise direction. The value for the counterclockwise direction is its negative, so the value of the integral over the circle is

$$(-4+2\pi i) - (-4-2\pi i) = 4\pi i.$$

In many cases, the value of a contour integral depends only on the endpoints rather than on the specific path. If $f(z)$ is analytic in the whole region between two paths which have the same endpoints, then the integral of $f(z)$ over either path is the same. This often allows us to calculate integrals over extremely complicated contours simply by replacing the the complicated contour by simple contours such as a portion of a circle or a pair of straight lines.

The basic idea is that given a function to integrate and a contour to integrate it over, we are allowed to deform that contour any way we like provided that it doesn't pass through any singularities in the deformation process. For example, consider the function $f(z) = \frac{1}{z}$, which we are asked to integrate over the contour C_1 shown in the graphic below. This is a difficult (and maybe impossible) curve to parametrize, however, we get the same value for the integral if we integrate over the contour C_2 which is very easy to parametrize. We can do this because the only singularity of $f(z)$ occurs at $(0, 0)$ and, so $f(z)$ is analytic in the entire region between the two curves. That is to say that C_1 can be deformed into



C_2 without crossing a singularity.

4.3 Cauchy's Theorem

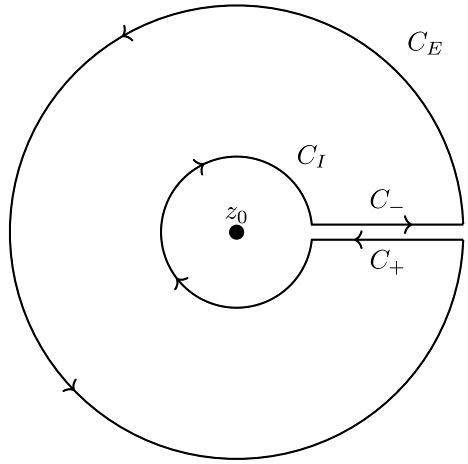
A **simply connected region** is a region in which every closed curve can be shrunk continously to a point within the region. That is, a simply connected region is a region with no holes. A **multiply-connected region** is a region containing multiple holes. An n-connected region is one containing n holes.

The **Cauchy theorem** states that if $f(z)$ is analytic at all points of a simply connected region, and if C is a simple (i.e. non-intersecting) closed contour within that region, then

$$\oint_C f(z)\, dz = 0.$$

Another way to state is that if $f(z)$ is analytic on and within a simple closed curve C, then the above is true. Cauchy's theorem is a direct result of the path independence discussed above. Cauchy's theorem can be derived by applying **Green's theorem** to complex functions.

What if a complex function $f(z)$ is analytic in a region except at one point z_0 (or instead a hole centered at z_0)? If this is the case, then we know that $f(z)$ is analytic in an annulus around z_0. By cutting the annulus, we can create a closed contour around every point in the region except for z_0, as in the image below. The contour integral along the path is the line integral along the exterior of the contour C_E plus the line integral along the interior contour C_I, plus the contributions of the line integrals along C_- and C_+.



Since the lines C_- and C_+ are very close to each other, the function is essentially the same along each path, and so the line integral along each path is essentially the same. In fact, as the distance between the two paths goes to zero, the line integrals along them become identical. However, since we integrate in opposite directions along the two paths, the two integrals cancel each other out, and so the contributions from C_+ and C_+ sum to zero.

By the Cauchy integral theorem, we know that the total contour integral is zero, since the annular path encloses a simply connected region wherein $f(z)$ is analytic. That is, the point z_0 is not *inside* the annular path.

$$\begin{aligned}
\int_C &= \int_{C_E} + \int_{C_+} + \int_{C_I} + \int_{C_-} = 0 \\
&= \oint_{C_E} + \oint_{C_I} = 0 \\
&= \oint_{C_E} - \oint_{C_I} = 0 \\
\oint_{C_E} &= \oint_{C_I} .
\end{aligned}$$

We are allowed to bend the path in this way because the integral of an analytic function over a closed path has the same value after deformation of the closed path as long as the path stays within the region of analyticity.

This leads us to **Cauchy's generalized theorem**: If $f(z)$ is analytic on an n-connected region with an outside boundary curve C_0 and inside boundary curves (i.e. around singular points or holes) $C_1, \ldots, C_n$, then

$$\int_{C_0} f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz.$$

That is, the integral around the exterior boundary is the sum of the integrals of the interior boundaries. Note: All the curves must be oriented in the same (counterclockwise) direction.

Example 4.3.1

Find

$$\oint_C z^n\,dz,$$

where C is the circle around the origin with radius r.

Our first step is to parametrize the circle C so that we can turn the contour integral into ordinary integrals. We can parametrize the circle as $z = re^{i\theta}$ where θ goes from 0 to 2π. Then $dz = ire^{i\theta}d\theta$.

Recall that $\frac{d}{dz}z^n = nz^{n-1}$, so it follows that the integral of z^n is $\frac{z^{n+1}}{n+1}$ except where $n = -1$, which would cause division by zero.

To evaluate this integral we break it into two parts, the case where $n \neq -1$ and the case where $n = -1$. When $n \neq -1$, we have that

$$\begin{aligned}\oint_C z^n\,dz &= \int_0^{2\pi} (r^n e^{in\theta}) ire^{i\theta}\,d\theta \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta \\ &= r^{n+1}\left(\frac{e^{i(n+1)\theta}}{i(n+1)}\right)\Bigg|_0^{2\pi} \\ &= 0.\end{aligned}$$

We know that the quantity is zero because $e^{i\theta}$ is periodic with period 2π.

For the case $n = -1$, we have that

$$\begin{aligned}\oint_C \frac{1}{z}\,dz &= i\int_0^{2\pi} d\theta \\ &= 2\pi i.\end{aligned}$$

So we have that

$$\oint_C z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

Notice that for $n \geq 0$, $f(z) = z^n$ is analytic everywhere, so by the Cauchy theorem we know the integral must be zero. For $n < 0$, we can't use the Cauchy theorem because $f(z) = \frac{1}{z^n}$ has a singular point where $z = 0$.

4.4 Integral Bounds

Recall from calculus that

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt \leq M\,|b-a|\,,$$

where M is the maximum value of $|f(t)|$ in $[a, b]$. What this says is simply that the absolute value of an integral is less than or equal to the maximum value of the integrand times the length of the interval. If we think of the integral as the area under a positive curve, then $M|b - a|$ is the area of a rectangle with width $b - a$ and height M, which obviously encloses the the entire area under the curve from a to b and a little more unless the function is a constant. If it is constant, then the equality holds.

We can find a similar upper bound for contour integrals. If we parametrize the curve

C as $z = \gamma(t)$ then $dz = \gamma'(t)\,dt$ and

$$\begin{aligned} \left|\int_C f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \\ &\le M\int_a^b |\gamma'(t)|\,dt \\ &= ML(C). \end{aligned}$$

That is,

$$\left|\int_C f(z)\,dz\right| \le ML(C),$$

where M is the upper bound on $|f(z)|$ for $z \in \mathbb{C}$ and $L(C)$ is the length of the curve C.

So to find an upper bound on an integral, we have to compute the length of the curve $L(C)$ as well as the upper bound M of the integrand on that curve. If the integrand is a rational function, then we can work with the numerator and denominator separately. We want to find an upper bound for the numerator and a lower bound on the denominator, then when we take the reciprocal of the denominator, the lower bound becomes an upper bound.

To find upper or lower bounds of polynomials, we can often use the triangle inequality which states that

$$|z + w| \le |z| + |w|.$$

Example 4.4.1

Find an upper bound for the integral

$$\int_C \frac{\operatorname{Log} z}{z+5}dz,$$

where C is the circle of radius $R > 1$ centered at the origin.

Since C is a circle of radius R, we know that $C(L) = 2\pi R$. To find an upper bound of the integrand, we find an upper bound for the numerator and a lower bound for the denominator and multiply. The numerator, we can write as

$$\operatorname{Log} z = \ln|z| + i\ \operatorname{Arg}\ z.$$

By the triangle inequality, then

$$|\operatorname{Log} z| \le |\ln|z|| + |i\ \operatorname{Arg}\ z|.$$

We know that $|z| = R$ and that $R > 1$ which means $\ln R > 0$ and so $|\ln R| = \ln R$.

We also know that $|i| = 1$ and that $-\pi <\ \operatorname{Arg}\ z \le \pi$ which means $|\operatorname{Arg}\ z| \le \pi$. Therefore,

$$|\operatorname{Log} z| \le \ln R + \pi.$$

Next, we look at the denominator. We can write $|z| = |z + 5 - 5|$. By the triangle inequality, we have that $|z + 5 - 5| \le |z + 5| + |-5|$, which simplifies to $|z+5| \ge |z| - 5$. Since $|z| = R$, we can write $|z+5| \ge R-5$. Taking the reciprocal gives us

$$\frac{1}{|z+5|} \le \frac{1}{R-5}.$$

Now we know the upper bound on the integrand is

$$\left|\frac{\operatorname{Log} z}{z+5}\right| \leq \frac{\ln R + \pi}{R-5} = M.$$

Remembering to multiply by $C(L)$, we have that the upper bound on the integral is

$$\left|\int_C \frac{\operatorname{Log} z}{z+5} dz\right| \leq 2\pi R \frac{\ln R + \pi}{R-5}.$$

Example 4.4.2

Find an upper bound for the integral

$$\int_C \frac{z^2+1}{z^3-z^2} dz,$$

where C is the upper semicircle of radius $R > 1$ centered at the origin.

We know that the length of the curve is $L(C) = \pi R$. Next, looking at the numerator and applying the triangle inequality, we find that

$$|z^2+1| \leq |z^2| + |1| = R^2 + 1,$$

since $|z| = R$. Factoring the denominator, we get $z^3 - z^2 = z^2(z-1)$. We know that $|z^2| = |z|^2 = R^2$. We can write $|z| = |z-1+1|$ then by the triangle inequality, we know that $|z-1+1| \leq |z-1| + |1|$, which implies that

$$|z-1| \geq |z| - 1 = R - 1.$$

or

$$\frac{1}{|z-1|} \leq \frac{1}{R-1}.$$

So for the integrand, we have that

$$\left|\frac{z^2+1}{z^3-z^2}\right| = \left|\frac{z^2+1}{z^2(z-1)}\right| \leq \frac{R^2+1}{R^2(R-1)}.$$

Putting it all together we have that

$$\left|\int_C \frac{z^2+1}{z^3-z^2} dz\right| \leq \pi \frac{R^2+1}{R(R-1)}.$$

4.5 Cauchy Integral Formula

Consider the integral

$$\oint_C (z-z_0)^n \, dz,$$

where C is any closed path surrounding z_0. We can shrink the path C to a circle around z_0 without changing the value of the integral, so by a simple substitution $w = z - z_0$, this integral becomes the same as the one done in an example earlier

$$\oint_C (z-z_0)^n \, dz = \oint_C w^n \, dw = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases},$$

for any counterclockwise path C that encloses z_0. In particular,

$$\oint_C \frac{1}{z - z_0}\, dz = 2\pi i. \tag{4.1}$$

Now consider the contour integral

$$\oint_C \frac{f(z)}{z - z_0}\, dz,$$

where $f(z)$ is analytic on and within a closed contour C, and z_0 is any point inside the region bounded by C. Notice that $\frac{f(z)}{z-z_0}$ is not analytic at $z = z_0$. We showed earlier that we can deform C into a circle about z_0 and then we can parametrize the circle as $z = z_0 + re^{i\theta}$ then $dz = ire^{i\theta} d\theta$ and θ goes from 0 to 2π. So our integral becomes

$$\begin{aligned}\oint_C \frac{f(z)}{z - z_0}\, dz &= \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(z_0 + re^{i\theta}) - z_0} ire^{i\theta}\, d\theta \\ &= i \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.\end{aligned}$$

Deforming C further, we shrink the circle, letting $r \to 0$, then if $f(z)$ is analytic (i.e. continuous) at z_0,

$$i \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \to i \int_0^{2\pi} f(z_0)\, d\theta = 2\pi i f(z_0).$$

If z_0 is any point outside the closed contour C, then we know that the integrand is analytic on and within C, and by Cauchy's theorem the integral is zero. This gives us **Cauchy's integral formula**:

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz = \begin{cases} f(z_0) & z_0 \text{ is within } C \\ 0 & z_0 \text{ is outside of } C \end{cases}$$

It is often written as follows. If $f(z)$ is analytic on and within a simple closed curve C containing the point z_0, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz.$$

To use Cauchy's integral formula to solve contour integrals, we try to write the integrand in the form $\frac{f(z)}{z-z_0}$ where z_0 is inside the the contour. Then the solution is just $2\pi i\, f(z_0)$. We need a linear term in the denominator, and everything else becomes part of $f(z)$. Note that $f(z)$ has to be analytic inside the contour. If the term in the denominator is not linear or if the coefficient on z is not 1, then we have to factor the denominator in order to get it in the right form.

Example 4.5.1

Evaluate

$$I = \oint_C \frac{1}{z(z+2)} dz,$$

where C is the unit circle with radius 1 and centered on the origin.

Notice that we can write

$$\frac{1}{z(z+2)} = \frac{1}{(z+2)(z-0)} = \frac{\frac{1}{z+2}}{z-0}.$$

Now we have the integrand in the form $\frac{f(z)}{z-z_0}$ where $f(z) = \frac{1}{z+2}$ and $z_0 = 0$.

By the Cauchy integral theorem, the solution is

$$I = 2\pi i f(0) = 2\pi i \frac{1}{0+2} = \pi i.$$

Tip

When looking for the singularities of a function, don't forget to include the complex roots of the denominator.

Example 4.5.2

Evaluate

$$I = \oint_C \frac{1}{4z^2 - 1} dz,$$

where C is the unit circle with radius 1 and centered on the origin.

We can write the denominator as $4z^2 - 1 = 4(z + \frac{1}{2})(z - \frac{1}{2})$, and then using partial fraction decomposition, we can rewrite the integrand as

$$\frac{1}{4z^2 - 1} = \frac{1}{4}\left(\frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}}\right).$$

We can now write our integral as

$$I = \frac{1}{4}\oint_C \frac{1}{z - \frac{1}{2}} dz - \frac{1}{4}\oint_C \frac{1}{z + \frac{1}{2}} dz,$$

where the integrands are both in the form $\frac{f(z)}{z-z_0}$ with $f(z) = 1$ and $z_0 = \frac{1}{2}$ for the first one and $z_0 = -\frac{1}{2}$ for the second one. Since both z_0 are within C, we can use Cauchy's integral formula, and our solution is

$$\begin{aligned} I &= \frac{1}{4}\left[2\pi i f\left(\frac{1}{2}\right) - 2\pi i f\left(-\frac{1}{2}\right)\right] \\ &= \frac{1}{4}[2\pi i \cdot 1 - 2\pi i \cdot 1] \\ &= 0. \end{aligned}$$

The Cauchy integral formula allows only one singularity inside the contour. However, if there is more than one, we can use the generalized Cauchy theorem, to write the integral on the contour as the sum of integrals on contours around each of the singularities. That is, we can break the problem into pieces, by replacing the single contour containing multiple singularities with multiple contours—one around each singularity. Then we can apply the Cauchy integral formula to each of them and sum the results to get the final answer. If there are additional singularities outside the original contour, then they are irrelevent and can be ignored.

From Cauchy's integral formula, we know that

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0} dz.$$

provided that z_0 is inside C. We also know that the derivative of $f(z_0)$ is

$$f'(z_0) = \frac{df}{dz_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

Tip

For the Cauchy integral formula, the contour must be oriented counterclockwise. Otherwise, the sign must be changed.

so we have that the derivative of $f(z_0)$ can be calculated as

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0}\frac{1}{h}\frac{1}{2\pi i}\left[\oint_C \frac{f(z)}{z-(z_0+h)}\,dz - \oint_C \frac{f(z)}{z-z_0}\,dz\right] \\ &= \lim_{h\to 0}\frac{1}{h}\frac{1}{2\pi i}\oint_C\left[\frac{f(z)}{z-(z_0+h)} - \frac{f(z)}{z-z_0}\right]dz \\ &= \lim_{h\to 0}\frac{1}{h}\frac{1}{2\pi i}\oint_C f(z)\left[\frac{(z-z_0)-(z-z_0-h)}{(z-z_0-h)(z-z_0)}\right]dz \\ &= \lim_{h\to 0}\frac{1}{h}\frac{1}{2\pi i}\oint_C f(z)\left[\frac{h}{(z-z_0-h)(z-z_0)}\right]dz \\ &= \lim_{h\to 0}\frac{1}{2\pi i}\oint_C f(z)\left[\frac{1}{(z-z_0-h)(z-z_0)}\right]dz \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz. \end{aligned}$$

We can repeat this process—differentiating the result above again using the same process, and we find that the nth derivative of $f(z_0)$ is

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

provided that $f(z_0)$ is an analytic function, regular (i.e. single-valued and differentiable) within the closed contour C, and continuous on C. Then the derivatives of all orders are also regular within C. So given the value of the integral of an analytic function over any curve C about the point z_0, we know the value of the function at that point as well as the value of every derivative at that point.

This tells us that if the linear term in the denominator is raised to a power, then instead of using the standard Cauchy integral formula, we have to use the more general derivative form of it given above.

Morera's theorem states that if

$$\oint_C f(z)\,dz = 0,$$

for every closed curve C within a region R in which $f(z)$ is continuous, then $f(z)$ is analytic in R.

Example 4.5.3

If C is any simple closed contour, find the value of

$$g(z) = \oint_C \frac{s^2+s}{(s-z)^2}ds.$$

Notice that this defines a function of z. Depending on the contour and z, this function will give different values. Notice that z takes the place of z_0 in the Cauchy integral formula and s takes the part of z.

We have to consider two cases: when z is inside C and when z is outside C. When z is outside z, then by the Cauchy theorem, the integral evaluates to zero. When z is inside C, we have to use the Cauchy integral formula. Since the linear term in this case is raised to a power, we have to use the derivative form of the Cauchy integral formula. In this case, $z_0 = z$, $f(s) = s^2 + s$, and $n = 1$, so

$$g(z) = \oint_C \frac{s^2+s}{(s-z)^2}ds = 2\pi i\, f'(z).$$

Differentiating $f(s)$ and plugging in z we find that $f'(z) = 2z + 1$. So our final answer is

$$g(z) = \begin{cases} (2z+1)2\pi i & \text{if } z \text{ is inside } C \\ 0 & \text{if } z \text{ is outside } C. \end{cases}$$

4.6 Summary: Integration

Integration

There are two kinds of complex integrals.

If the variable is real, then you have a regular integral. If the differential is $d\theta$ or dt, then the variable is real. There are two ways to evaluate such an integral.

1) Write the integrand in the form $f = u + iv$, integrate u and v separately and add the results, multiplying the second integral by i. That is, if $f(t) = u(t) + iv(t)$, then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

2) Integrate directly, treating any complex number as a constant. This is obviously the easiest way.

If the integral is complex, for example, the differential is dz, then you have to integrate over a contour. There are again several options depending on what integral you're dealing with.

1) This method always works, provided that you can parametrize the contour. First, parametrize the contour C to as a function of t where $a \le t \le b$, then make the substitution $z = C(t)$ so that $dz = C'(t)\,dt$, then the integral becomes

$$\int_C f(z)\,dz = \int_a^b f(C(t))C'(t)\,dt.$$

We can parametrize a circle of radius r centered at z_0 as

$$z = C(t) = z_0 + re^{it}.$$

We can parametrize a line segment running from a point z_0 to a point z_1, as

$$z = C(t) = (1-t)z_0 + tz_1.$$

When integrating over a contour, remember that direction matters. The integral of a function over a contour in the clockwise direction is the negative of the integral over the contour in the counterclockwise direction. We can also add contours together to form a longer contour. For example, if the path is composed of multiple straight line segments, we just break the integral into as many parts as there are line segments, compute each integral separately using the parametrization for that specific line segment, then add the results together to get the value of the integral over all the line segments. When adding contours, the direction must be the same for all of them (i.e. all counterclockwise or all clockwise).

2) If the function is analytic (at least in some region), then we can deform the contour in any way provided that we don't push the contour through any singularity of the function and provided that the endpoints of the contour remain fixed without changing the value of the integral. In other words, we can replace a complicated contour with a simpler contour (with the same endpoints) provided that the function is analytic between the two contours.

3) If the contour is a simple, closed curve, and if the function is analytic on and within the curve, then by Cauchy's theorem, the value of the integral is zero

$$\oint_C f(z)\,dz = 0.$$

4) If the contour is a simple, closed curve C_0, and if the function is analytic on and within C_0 except at a finite number of singularities or holes with boundaries $C_1, \ldots, C_n$, then by the generalized Cauchy theorem, the integral over the external contour is the sum of the integrals over the internal contours

$$\oint_{C_0} f(z)\,dz = \sum_{i=1}^{n}\oint_{C_i} f(z)\,dz,$$

provided that all contours have the same direction.

5) If the contour is closed and the integrand is analytic on and within the contour, and if z_0 is any point inside the contour, if we can rewrite the integrand in the form $\frac{f(z)}{z-z_0}$ then the value of the integral is $2\pi i$ times $f(z)$ evaluated at the point z_0. This is Cauchy's integral formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

If the linear term in the denominator is raised to a power use the more general (derivative) form of the Cauchy integral formula

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0).$$

Integral Bounds

An upper bound on an integral is obtained as

$$\left|\int_C f(z)\,dz\right| \le ML(C),$$

where M is the upper bound on $|f(z)|$ for $z \in C$ and $L(C)$ is the length of the curve C.

The upper bound M of the integrand is often obtained with the help of the triangle inequality

$$|z+w| \le |z| + |w|,$$

and knowledge of the value of $|z|$. If the integrand is a rational function then we seek an upper bound P of the numerator and a lower bound Q of the denominator then $M = \frac{P}{Q}$.

Chapter 5

Series

5.1 Sequences

An infinite sequence of complex numbers $z_1, z_2, z_3, \ldots z_n$ has a limit z if for every radius ε, there exists an integer $N > 0$ such that $|z_n - z| < \varepsilon$ whenever $n > N$. That is, given an arbitrary disk neighborhood of z, if it is the case that when n is larger than some N then all z_n are in the disk neighborhood, and if this is true for all disk radii, then the sequence is converging to z. If the limit exists, we say it is a **convergent sequence** and we write

$$\lim_{n\to\infty} z_n = z.$$

Otherwise, we say it is a **divergent sequence**.

We can consider the real and imaginary parts of a sequence separately. That is, if $z_n = x_n + iy_n$ and $z = x + iy$, then $\lim_{n\to\infty} z_n = z = x + iy$ is equivalent to

$$\lim_{n\to\infty} x_n = x, \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y.$$

Example 5.1.1

Suppose we have a sequence of complex numbers in which the nth term is given by

$$z_n = 2 + \frac{3}{n} + ie^{-3n}.$$

Compute the limit of the sequence.

Separating the real and imaginary parts, we get

$$\lim_{n\to\infty} \left(2 + \frac{3}{n}\right) = 2,$$

and

$$\lim_{n\to\infty} e^{-3n} = 0.$$

Therefore,

$$\lim_{n\to\infty} z_n = 2.$$

5.2 Series

An infinite series

$$\sum_{n=1}^{\infty} z_n,$$

converges to the sum S if the sequence of partial sums

$$S_n = \sum_{n=1}^{\infty} z_n,$$

converges to S. If the sequence/series does not converge, then we say it diverges.

As with sequences, we can decompose a series into its real and imaginary parts. So if $z_n = x_n + iy_n$ and $S = a + bi$, then

$$\sum_{n=1}^{\infty} z_n = S,$$

is equivalent to

$$\sum_{n=1}^{\infty} x_n = a \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n = b,$$

and

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n.$$

Since we can decompose a complex series into a pair of series of real numbers (one of them multiplied by i), all the series rules that we are used to from calculus of real variables hold true for complex series. For example, if a series converges, then the terms in the series tend to zero as $n \to \infty$.

A complex series $\sum_{n=1}^{\infty} z_n$ is **absolutely convergent** if

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2},$$

converges. Note: An absolutely convergent series is also a convergent series.

An important series in both real and complex variables is the **geometric series**. The complex version is

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \text{for } |z| < 1.$$

This can be verified by starting with the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad \text{for } z \neq 1,$$

which gives us the Nth partial sum

$$S_N = \sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}.$$

Taking the limit of this as $N \to \infty$, we see that it converges to $\frac{1}{1-z}$ iff $|z| < 1$.

5.3 Taylor Series

Weierstrass' analytic continuation extended the definition of a function of a real variable by replacing it with $z = x + iy$. In general, a complex function $f(z)$ can be expressed as the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $f(z)$ is the **analytic continuation** of $f(x)$ into the complex plane.

There is a unique power series expansion for a complex function $f(z)$, and it is called the **Taylor series**. If $f(z)$ is analytic on and within the disk $|z - z_0| < R$, then it has the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{for } |z - z_0| < R,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Notice that this is the same definition of Taylor series as given in calculus of real variables, but now the variable is complex. From the generalized Cauchy integral formula, we also have that

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

As long as you choose z on or within the disk of analyticity then then the power series will agree exactly with $f(z)$. If you choose a z outside the disk of analyticity, then the two will not agree.

If $f(z)$ is analytic at a point z_0, then by definition, it is analytic in a small disk around z_0, and therefore, it has a Taylor series about z_0.

If $f(z)$ is an entire function, then the radius of convergence of its Taylor series is arbitrarily large. In other words, the Taylor series converges to $f(z)$ for every $z \in \mathbb{C}$.

The Taylor series for the complex exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \text{for } |z| < \infty.$$

Since the exponential function is entire, we just took the Maclaurin series since the same series can be used everywhere.

Example 5.3.1

Find the Taylor series for $f(z) = 4z^5 e^{6z}$.

To obtain the Taylor series, we don't even have to use the definition. We can just replace z by $6z$ in the Taylor series for e^z and then multiply it by $4z^5$ to get

$$\begin{aligned} 4z^5 e^{6z} &= 4z^5 \sum_{n=0}^{\infty} \frac{(6z)^n}{n!} \\ &= 4 \sum_{n=0}^{\infty} \frac{6^n z^{n+5}}{n!} \\ &= 4 \sum_{n=5}^{\infty} \frac{6^{n-5}}{(n-5)!} z^n. \end{aligned}$$

Using the definitions

$$\begin{aligned} \sin z &= \frac{1}{2i}(e^z - e^{-z}) \\ \cos z &= \frac{1}{2}(e^z + e^z), \end{aligned}$$

we can, after some rearranging and reindexing, obtain the Taylor expansions of $\sin z$ and $\cos z$ using the Taylor series for the exponential function.

$$\begin{aligned}\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.\end{aligned}$$

Since $\sin z$ and $\cos z$ are both entire functions, these Taylor series are valid for all $|z| < \infty$.

Since Taylor series are unique and the geometric series (given in the previous section) has the form of a Taylor series, it must be the Taylor series for $\frac{1}{1-z}$. By replacing z with $-z$ in the geometric series, we get

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} z^n(-z)^n, \qquad \text{for } |z| < 1.$$

By replacing z with $1-z$ in the geometric series, we get

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n, \qquad \text{for } |1-z| < 1.$$

Recall that for a real power series $\sum_{n=0}^{\infty} a_n x^n$, if it converges for $x = x_0$, then it converges absolutely for $|x| < |x_0|$ and uniformly for $|x| \le |x_1| < |x_0|$, where x_1 is some fixed value. If the series diverges for $x = x_0$ then it diverges for $|x| > |x_0|$.

For complex power series, absolute convergence of the real power series in the interval $|x| < R$ implies the absolute convergence of the analytic continuation $\sum a_n z^n$ inside the circle on the complex plane $|z| < R$, where R is the **radius of convergence**, and the circle is the **domain of convergence**.

The largest region of convergence is some circle of finite or infinite radius centered at $z = 0$. When R_{max} is finite and $f(z) = \sum a_n z^n$, then $f(z)$ has at least one **singularity** on the largest circle of convergence $|z| = R_{max}$. The largest circle of convergence of the power series $f(z) = \sum a_n z^n$ passes through the singularity of $f(z)$ that is closest to the origin.

A power series can be differentiated or integrated term-by-term inside any part of the region of convergence.

If we have a function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ that is convergent for $|z| < R$, then we may be able to extend the convergence beyond that little circle, by Taylor expanding about the point $x = a$ inside the original circle. Then $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is convergent for any circle around a that is within the original circle, but it may also be convergent beyond that. If it is, we get a new region of convergence and we can repeat this process until there are no more possible regions of existence. This is then the **natural boundary** of the function.

The set of all power series obtainable from an initial power series by analytic continuation is termed a **complete analytic function**. Each individual power series is considered an element of the complete function.

According to Cauchy's integral formula, an analytic function can be expressed as an integral. If a complex function is known on the boundary of a region, the function in the interior of the region is determined. Contrast this with the behavior of a real function. The value of a real function can be specified on any interval with no constraints on values of the function outside of the interval apart from continuity and its derivative.

Recall that

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

are the coefficients in a Taylor series. This gives us a formula for the nth derivative of some function $f(z)$ evaluated at a point z_0

$$f^{(n)}(z_0) = a_n n!.$$

In some cases, finding the Taylor series of a function from known Taylor series is a lot easier than directly finding a formula for the nth derivative of a function evaluated at a point. So if we can find the Taylor series for $f(z)$ about z_0, then we automatically get a formula for the nth derivative of $f(z)$ evaluated at z_0.

Example 5.3.2

Find the 30th derivative of $f(z) = (z-i)^2 e^{6z}$ evaluated at $z = i$.

We are looking for a series centered at i, so we know that our series will involve powers of $(z-i)^n$. The $(z-i)^2$ part of f is not a problem since the powers are of the right form. However, e^{6z} is not of the right form, so we have to manipulate it a little. Notice that we can write

$$\begin{aligned} f(z) &= (z-i)^2 e^{6z-6i+6i} \\ &= (z-i)^2 e^{6(z-i)+6i} \\ &= (z-i)^2 e^{6i} e^{6(z-i)}. \end{aligned}$$

Now all the parts have the right form, and we can find the Taylor series by starting with the Taylor series of the exponential function. Since the Taylor series of the exponential form is valid for any z, we can replace the z by $6(z-i)$ in the Taylor series of e^z.

$$\begin{aligned} f(z) &= (z-i)^2 e^{6i} \sum_{n=0}^{\infty} \frac{1}{n!}(6(z-i))^n \\ &= (z-i)^2 e^{6i} \sum_{n=0}^{\infty} \frac{6^n}{n!}(z-i)^n \\ &= \sum_{n=0}^{\infty} \frac{6^n e^{6i}}{n!}(z-i)^{n+2} \\ &= \sum_{n=2}^{\infty} \frac{6^{n-2} e^{6i}}{(n-2)!}(z-i)^n. \end{aligned}$$

This is now the Taylor series of $(z-i)^2 e^{6z}$ centered at $z = i$. From the formula for Taylor series, we know that the Taylor series for a function centered at i is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(i)}{n!}(z-i)^n.$$

This tells us that

$$\frac{f^{(n)}(i)}{n!} = \frac{6^{n-2} e^{6i}}{(n-2)!}.$$

Simplifying gives us

$$f^{(n)}(i) = n(n-1)6^{n-2} e^{6i}.$$

So the 30th derivative of f evaluated at i is

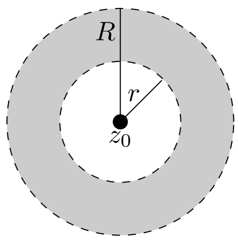
$$f^{(30)}(i) = 30 \cdot 29 \cdot 6^{28} \cdot e^{6i}.$$

5.4 Laurent Series

If we have $0 \le r < R \le \infty$, we define the notation

$$\text{Ann}(z_0, r, R),$$

as meaning the region $\{z \in \mathbb{C} : r < |z - z_0) < R\}$. That is, it is the open and connected annular region centered on z_0 with inner radius r and outer radius R.



Note, $\text{Ann}(z_0, 0, R)$ is a perfectly good annular region. It has an inner radius of 0, although the central point is not included, so this kind of region is also called the punctured R-neighborhood of z_0, or $N_R^*(z_0)$. We could also define $\text{Ann}(z_0, r, \infty)$, which is all complex numbers except a disk of radius r centered at z_0. We could even define $\text{Ann}(0, 0, \infty)$ which is also denoted $\mathbb{C}^*$ and called the punctured complex plane since it includes all complex numbers except $z = 0$.

Recall that for a function to have a Taylor series, it must be analytic on a disk. Some functions aren't analytic on a disk at certain places. For example, $f(z) = \frac{1}{z}$ is not analytic at $z = 0$, so there is no Taylor series for it centered at $z = 0$. However, there is a new kind of series that we can use.

Laurent's theorem tells us that given $z_0 \in C$ and $0 \le r < R \le \infty$, if $f(z)$ is analytic in $\text{Ann}(z_0, r, R)$, then there exist a_n and b_n in $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

for all $z \in \text{Ann}(z_0, r, R)$. This is called the **Laurent series** expansion of $f(z)$. The coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} dw \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{-n+1}} dw. \end{aligned}$$

The curve C is any simple closed curve entirely within the annular region that encloses the center point. Notice that the only difference between the integral formulas for a_n and b_n is the negative sign on n. Written in this way, the a_n part of the series is often called the **Taylor series part** and the b_n part is called the **principal part** of the Laurent series.

Observe that

$$b_1 = \frac{1}{2\pi i} \int_C f(w)\, dw.$$

This quantity is called the **residue** of f at z_0.

Another common way of representing the Laurent series is as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

Notice that this is different from earlier series we've encountered in that the index runs from $-\infty$ to ∞ instead of from 0 or 1 to ∞.

If $f(z)$ is analytic on and within this annular region then $b_n = 0$ for all n, and the Laurent series reduces to the Taylor series for $f(z)$. If $f(z)$ has a singularity at z_0 (i.e. the center of the annular region), then the a_n coefficients are no longer equal to the Taylor coefficients.

In general, we want to be able to find Laurent series without actually doing the integrals given above. A key fact is that if a series in the form of a Laurent series is found for a function, then the series is *the* Laurent series for the function. That is, Laurent series are unique. This allows us to often find Laurent series by manipulating known Taylor series. Also, if we find a Taylor series for a function then it must also be the Laurent series for the function.

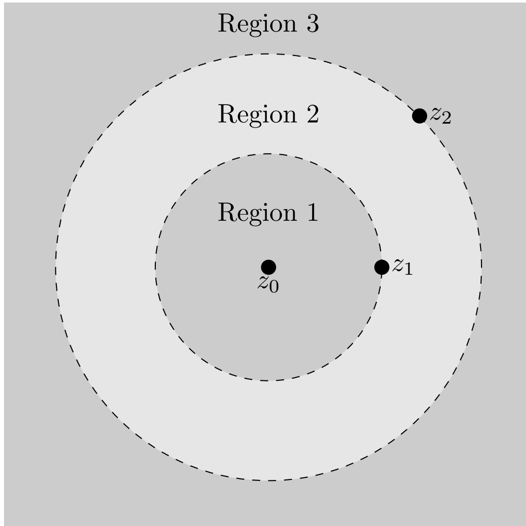
Example 5.4.1

Notice that $f(z) = e^{\frac{1}{z}}$ is analytic everywhere except at $z = 0$. This means we cannot find a Taylor series for $f(z)$ at $z = 0$. However, if we replace z by $\frac{1}{z}$ in the Taylor expansion of e^z, we get

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

This is not a Taylor series, but it *is* a valid series for all $z \in C$ except for $z = 0$ (i.e. $f(z)$ is analytic on $\text{Ann}(0, 0, \infty)$). Since the powers of z are negative, this is in the form of a Laurent series, and so it *is* the Laurent series for $e^{\frac{1}{z}}$. All the a_n terms are zero.

We know that if $f(z)$ has no singularities, then its Laurent expansion is the same as its Taylor expansion and is valid for all $\mathbb{C}$. If it has a single singularity at z_0, it has a Laurent expansion about z_0 and is valid in $\text{Ann}(z_0, 0, \infty)$. That is, the Laurent expansion is valid for all $z \neq z_0$. What happens if $f(z)$ has multiple singularities z_0, z_1, and z_2? If we center the Laurent expansion on z_0, then we know that $f(z)$ is analytic in the punctured disk (which is an annular region) extending out to the next closest singularity z_1. Therefore, the Laurent series we find for that region is only valid for that punctured disk. Then we have another annular region extending from an inner radius equal to the distance between z_1 and z_0 with an outer radius extending to the next closest singularity z_2, and so on. Each annular region on which $f(z)$ is analytic on will have its own unique Laurent series. So if $f(z)$ has n singularities, then it will have as much as $n + 1$ different annular regions of analyticity and $n + 1$ different Laurent series. If the Laurent series is centered on one of the singularities, then $f(z)$ may be described by only n Laurent series.



Numbers on the dashed circles, that is, the numbers on the boundary of an annular region of analyticity cause a problem in Laurent series. Sometimes the series diverges if we plug in a z value on the circle. Other times, the series may diverge, and it may even diverge to the wrong value. In general, to obtain a Laurent series valid for such a number, we have to center the series at a new location so that the number does not fall on the boundary of analyticity.

Following are some summation formulas that are useful when calculating Laurent series. If $|a| > |b|$, then $\left|\frac{b}{a}\right| < 1$ and

$$\frac{1}{a-b} = \frac{1}{a} \cdot \frac{1}{1-\frac{b}{a}}.$$

For geometric series, we know that

$$\frac{1}{a} \cdot \sum_{n=0}^{\infty} \left(\frac{b}{a}\right)^n = \frac{1}{a} \cdot \frac{1}{1-\frac{b}{a}},$$

so we have the useful formula

$$\frac{1}{a-b} = \sum_{n=0}^{\infty} \frac{b^n}{a^{n+1}} = \sum_{n=-1}^{-\infty} \frac{a^n}{b^{n+1}}. \tag{5.1}$$

Similarly, if $|a| < |b|$, then

$$\frac{1}{a-b} = -\sum_{n=0}^{\infty} \frac{a^n}{b^{n+1}} = -\sum_{n=-1}^{-\infty} \frac{b^n}{a^{n+1}}. \tag{5.2}$$

If $|a| > |b|$, then

$$\frac{1}{a+b} = \sum_{n=0}^{\infty} (-1)^n \frac{b^n}{a^{n+1}} = -\sum_{n=-1}^{-\infty} (-1)^n \frac{a^n}{b^{n+1}}. \tag{5.3}$$

and if $|a| < |b|$ then

$$\frac{1}{a+b} = \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{b^{n+1}} = -\sum_{n=-1}^{-\infty} (-1)^n \frac{b^n}{a^{n+1}}. \tag{5.4}$$

Note, given two absolutely convergent series, you can multiply them together (i.e. FOIL out the first several terms) to get the beginning of a third convergent series. Taylor and Laurent series are absolutely convergent, so you can multiply any two of them together in this manner.

Example 5.4.2

Find Laurent series expansions for

$$f(z) = \frac{1}{z+z^2}.$$

The first step is to identify the annular regions of analyticity for $f(z)$. To do that, we have to identify the singularities of $f(z)$. We can factor it as

$$f(z) = \frac{1}{z(z+1)},$$

so $f(z)$ has singularities at $z = 0$ and $z = 1$. If we center the Laurent series at $z = 0$, then our punctured disk of analyticity $\text{Ann}(0, 0, 1)$ extends out to $z = 1$. Outside of that, we have the annular region of analyticity $\text{Ann}(0, 1, \infty)$ extending out to infinity.

For $0 < |z| < 1$, we can use the geometric series to obtain the Laurent series.

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \frac{1}{1-(-z)} \\ &= \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-z)^n \\ &= \sum_{n=0}^{\infty} (-z)^{n-1} \\ &= -\frac{1}{z} + \sum_{n=0}^{\infty} (-1)^n z^n. \end{aligned}$$

This is a Laurent series for $f(z)$ with $a_n = (-1)^n$, $b_1 = -1$, and $b_n = 0$ for $n > 1$. Keep in mind that this series representation for $f(z)$ is only valid for z in the annular region $\text{Ann}(0, 0, 1)$.

Now we want to find the Laurent series for z outside of $\text{Ann}(0, 0, 1)$. We do that by factoring a z out of the denominator to get

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1-\left(-\frac{1}{z}\right)}.$$

In the region of interest, $|z| > 1$ which implies that $\frac{1}{|z|} < 1$ and so we can again use the geometric series since the key value is less than one.

$$\begin{aligned} f(z) &= \frac{1}{z^2} \cdot \frac{1}{1-\left(-\frac{1}{z}\right)} \\ &= \frac{1}{z^2} \cdot \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n z^{-n-2} \\ &= \sum_{n=2}^{\infty} (-1)^n z^{-n}. \end{aligned}$$

Now we have the Laurent series for $f(z)$ that is valid for $|z| > 1$.

Example 5.4.3

Find the Laurent expansion of

$$f(z) = \frac{1}{z(z-1)},$$

around $z_0 = 0$.

Notice that $f(z)$ has singularities at $z = 0$ and $z = 1$. If we let $r > 0$ and $R < 1$ then $0 < |z| < 1$. From the formula for Laurent expansion, we get

$$f(z) = \frac{1}{z(z-1)} = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Evaluating $f(z)$ at $z = z'$ and plugging this into the formula for a_n, we get

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{1}{z'(z'-1)} \frac{1}{(z'-z_0)^{n+1}} \, dz' \\ &= \frac{1}{2\pi i} \oint_C \frac{1}{(z')^{n+2}(z'-1)} \, dz'. \end{aligned}$$

We know that $|z'| < 1$ so by Eq. (5.2) we have that

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{1}{(z')^{n+2}} \left(-\sum_{m=0}^{\infty} (z')^m \right) dz' \\ &= -\frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint_C (z')^{m-n-2} \, dz' \end{aligned}$$

We showed earlier that $\oint z^n \, dz$ is $2\pi i$ if $n = -1$ and otherwise, it is 0, so the integral above is $2\pi i$ if $m = n + 1$, and otherwise, it is zero. We can write this result as

$$\begin{aligned} a_n &= -\sum_{m=0}^{\infty} \delta_{m,n+1} \\ &= \begin{cases} -1 & n \geq -1 \\ 0 & n < -1 \end{cases}. \end{aligned}$$

So the Laurent expansion is

$$\frac{1}{z(z-1)} = -\sum_{n=-1}^{\infty} z^n = -\frac{1}{z} - 1 - z^2 - z^3 - \cdots.$$

We could also have found this another way by starting with the partial fraction decomposition

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} - \frac{1}{1-z}.$$

Since $|z| < 1$, we can expand the second term using Eq. (5.1).

$$\begin{aligned} \frac{1}{z(z-1)} &= -\frac{1}{z} - \sum_{n=0}^{\infty} z^n \\ &= -\sum_{n=-1}^{\infty} z^n. \end{aligned}$$

5.5 Summary: Series

The Taylor series for a complex function $f(z)$ analytic on and within the disk $|z - z_0| < R$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad \text{for } |z - z_0| < R.$$

The largest disk of convergence extends from the center of the disk at z_0 to the nearest singularity.

The Laurent series for a complex function $f(z)$ analytic in the annular region centered at z_0 with inner r and outer radius R is

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n},$$

for all $z \in \text{Ann}(z_0, r, R)$. The coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z_0)^{n+1}}dw \\ b_n &= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z_0)^{-n+1}}dw. \end{aligned}$$

The curve C is any simple closed curve entirely within the annular region that encloses the center point.

If a function has multiple singularities, then there will be multiple annular regions, each annular region having its own Laurent series.

Laurent series are unique, and all complex Taylor series are Laurent series (albeit with no negative powers.

Some important Taylor/Laurent series (centered at $z_0 = 0$) to remember are

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \text{for } |z| < \infty \\ \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad \text{for } |z| < \infty \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \qquad \text{for } |z| < \infty \\ \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n, \qquad \text{for } |z| < 1. \end{aligned}$$

The last one is the **geometric series**. Be sure and don't forget the $(-1)^n$ in the Taylor series for the sine and cosine functions.

To find the Taylor/Laurent series for a function $f(z)$ use the following procedure:

1. Identify z_0, the center of the disk or annular region. This will tell you the form of the series $\sum a_n(z - z_0)^n$.
2. Identify the singularities of $f(z)$.
3. Plot z_0 and all the singularities on the complex plane. This will allow you to identify the largest disk of convergence if you're dealing with a Taylor series and the different annular regions if you're working with Laurent series.
4. Find the series by manipulating known series. Remember, if you have multiple singularities and you're asked to find *all* the Laurent series, you will need a different Laurent series for each annular region.

To find the nth derivative of $f(z)$ evaluated at z_0, find the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where the nth coefficient is

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Once you have found the Taylor series, the nth coefficient is known, and you can solve for the nth derivative of $f(z)$ evaluated at z_0

$$f^{(n)}(z_0) = a_n\, n!$$

Chapter 6

Singularities and Residues

6.1 Isolated Singularities

If $f(z)$ is analytic in the punctured disk $N^*_\varepsilon(z_0)$, but $f'(z_0)$ does not exist, then z_0 is an **isolated singularity**. That is, it is the only singularity in some small disk neighborhood.

Singularities that are not isolated are much more difficult to work with.

There are three types of isolated singularities: removable singularities, poles, and essential singularities.

A **meromorphic function** is a function that is analytic in some region except at a set of isolated poles. A meromorphic function can be represented as a ratio of two analytic (regular and holomorphic) functions provided that the denominator is defined as being nonzero.

Removable Singularities

If z_0 is a removable singularity, it means we can redefine the function at z_0, so that it becomes analytic. A singularity is removable if $b_n = 0$ for all n in the Laurent expansion of $f(z)$ about z_0. In this case, the Laurent series reduces to the Taylor series

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Notice if we take the limit of the right side as $z \to z_0$, we get a_0. The same is true for the left side, and we can write

$$\lim_{z \to z_0} f(z) = a_0.$$

If $f(z)$, analytic on $N^*_R(z_0)$, has an isolated singularity at z_0 with $b_n = 0$ for all n, then we can remove the singularity by redefining $f(z)$ so that it is analytic on the whole disk. We redefine it as

$$f(z) = \begin{cases} \text{old } f(z), & 0 < |z - z_0| < R \\ a_0, & z = z_0. \end{cases}.$$

The a_0 can be found either by looking at the Laurent series or by taking the limit given above.

There are two ways we can determine if an isolated singularity z_0 is removable:

1. We can write the Laurent series for the function. If all $b_n = 0$, then the singularity is removable.
2. We can take the limit $\lim_{z \to z_0} f(z)$. If it exists, then it is a removable singularity.

Example 6.1.1

Determine the nature of the singularity at $z = 0$ of

$$f(z) = \frac{\sin z}{z},$$

and remove it if it is removable.

To show that z_0 is an isolated singularity, we note that it is made of two entire functions, which means it can be differentiated using the familiar quotient rule. Doing so, shows us that it is analytic everywhere except at $z = 0$. Therefore, $z = 0$ is an isolated singularity.

Usually the easiest way to check is to take the limit.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1.$$

Since the limit exists, it is a removable singularity.

The other way to check is to look at the Laurent series.

$$\begin{aligned} f(z) &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}. \end{aligned}$$

It is now in the form of a Laurent series, and since there are no b_n terms, z_0 must be a removable singularity.

We can redefine $f(z)$ as

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}.$$

Poles

The singularity z_0 is a **pole of order** m if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}.$$

In other words, if the Laurent series has only finite negative powers, then z_0 is a pole. The largest nonzero negative power is the order of the pole.

A **simple pole** is a pole of order $m = 1$.

This kind of singularity cannot be removed.

Notice that if we multiply the equation above by $(z - z_0)^m$, we get a Taylor series where all the powers are positive. The result is a series with a removable singularity at z_0. That is,

$$\phi(z) = (z - z_0)^m f(z),$$

has a removable singularity at z_0 if $f(z)$ has a pole of order m at z_0. Note, a key requirement is that $(z - z_0)^{m-1} f(z)$ does not have a removable singularity.

In other words, $f(z)$ has a pole of order m at z_0 if

$$\lim_{z\to z_0}(z-z_0)^m f(z),$$

exists, but

$$\lim_{z\to z_0}(z-z_0)^{m-1} f(z),$$

does not exist. So if we think that $f(z)$ has a pole of order m at z_0, we can confirm it by checking these two limits.

Tip

When identifying the singularities of a complex function, remember to include the complex zeros of the denominator—not just the real zeros.

Example 6.1.2

Characterize the singularity at $z = 3$ of

$$f(z) = \frac{e^z}{(z-3)^2}.$$

Since $f(z)$ is differentiable everywhere except $z = 3$, this is an isolated singularity. The Taylor series of e^z about $z = 3$ is

$$e^z = \sum_{n=0}^{\infty} \frac{e^3}{n!}(z-3)^n,$$

so the Laurent series of $f(z)$ about the same point is

$$\begin{aligned} f(z) &= \frac{1}{(z-3)^2}\sum_{n=0}^{\infty}\frac{e^3}{n!}(z-3)^n \\ &= \sum_{n=0}^{\infty}\frac{e^3}{n!}(z-3)^{n-2} \\ &= \frac{e^3}{(z-3)^2} + \frac{e^3}{z-3} + \sum_{n=2}^{\infty}\frac{e^3}{n!}(z-3)^{n-2}. \end{aligned}$$

We see that the largest negative power in the Laurent expansion of $f(z)$ is 2, so $f(z)$ has a pole of order 2 at $z = 3$.

Multiplying $f(z)$ by $(z-z_0)^2$ gives us a new function with a removable singularity at $z = 3$.

In general, if $f(z)$ is something divided by a polynomial, then the order of the pole is the same as the degree of the polynomial. If the polynomial is factorable, then then $f(z)$ has multiple singularities—one corresponding to each factor in the denominator, and the order of these poles are the powers that these factors are raised to.

We can also say that $f(z)$ has a pole of order n at $z = z_0$ if $g(z) = \frac{1}{f(z)}$ has a zero of order n at $z = z_0$. For example, $f(z) = \frac{1}{z^3}$ has a pole of order 3 at $z = 0$ because $g(z) = z^3$ has a simple zero of order 3 at $z = 0$.

Essential Singularities

An essential singularity occurs at z_0 if the Laurent expansion of $f(z)$ about z_0 has infinitely many nonzero b_n. Note, not all b_n have to be nonzero, but there have to be infinitely many of them such that there is no largest negative power m.

The function

$$f(z) = e^{\frac{1}{z}},$$

has an essential singularity at $z = 0$.

Example 6.1.3

Characterize the singularities of

$$f(z) = e^{\frac{1}{z}}.$$

We know there is a singularity at $z = 0$ because of the $\frac{1}{z}$. Expanding $f(z)$ around $z = 0$, we get

$$e^{\frac{1}{z}} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{6}z^{-3} + \cdots$$

Notice that every term except for the first term are actually the terms of the principle part of the Laurent expansion—see the negative powers on z. Since there are infinite nonzero terms in the principal part of the Laurent expansion, $z = 0$ is an *essential singularity*.

Example 6.1.4

Characterize

$$f(z) = \sin z$$

at $z = \infty$.

The Laurent/Taylor expansion of $\sin z$ is

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

To characterize $z = \infty$, we look at $z = \frac{1}{w}$. Since

$$\sin\left(\frac{1}{w}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{-(2k+1)},$$

has infinite terms in the principal part of the Laurent expansion, it has an essential singularity at $w = 0$, which implies that $\sin z$ has an essential singularity at $z = \infty$.

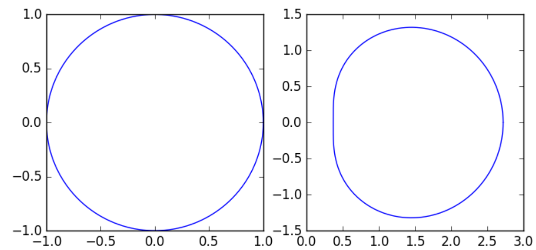
In general, to classify singularities using the Laurent series, we can use the following process. For a complex function $f(z)$ expanded around z_0, if z_0 is a singularity, then

- if all b_n are zero, then z_0 is a **removable singularity**.
- if only finitely many b_n are nonzero, then z_0 is a **pole of order** m where m is the largest n such that b_n is nonzero. In particular, if b_1 is nonzero, but all other b_n are zero, then z_0 is a **simple pole**.
- if infinitely many b_n are nonzero, then z_0 is an **essential singularity**.

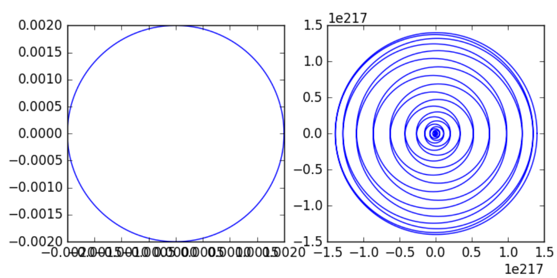
6.2 Casorati-Weierstrass Theorem

The **Casorati-Weierstrass theorem** states that if $f(z)$ has an isolated essential singularity at $z = z_0$, then for any δ and ε greater than zero and any complex a: $|f(z) - a| < \varepsilon$ for some z with $|z - z_0| < \delta$. In other words, in an arbitrary neighborhood of an isolated essential singularity, a function comes arbitrarily close to any complex number.

For example, if we choose a radius of $\varepsilon = 1$, and plot the circle $z = 1$ and its image on the w-plane, we get the following image:



On the left side we plot the circle $z = 1$ (i.e. the input or z-plane), and on the right side, we plot the image of the circle (i.e. the w-plane). If we now decrease the radius to $z = 0.002$, we get the following image:



Notice that as the circle got smaller, its image got much larger and much more complicated. Essentially what is happening is that as the radius of the circle becomes smaller and smaller, its image becomes a space filling curve, and that is how the function $e^{\frac{1}{z}}$ comes arbitrarily close to any complex number in an arbitrarily small neighborhood about its essential singularity at $z = 0$.

6.3 The Residue Theorem

If $f(z)$ is analytic inside a region enclosed by the closed path C except at n isolated singularities, $z_1, z_2, \ldots, z_n$ then the **residue theorem** states that

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res }(f, z_k).$$

That is, the integral of a function along a closed path C is $2\pi i$ times the sum of the residues of the singularities enclosed by C. When using this theorem, we don't care about any singularities that may appear outside of C.

Tip

Remember that b_1 is the *coefficient* of the first negative power in the Laurent series.

Tip

Keep in mind that to find the residue of a singularity z_0 using the Laurent series, the Laurent series has to be expanded about that point. If you have multiple singularities, you have to compute the same number of series to find the residues.

If $f(z)$ is analytic everywhere within and on a closed curve C surrounding a point z_0, then the **residue** of $f(z)$ at $z = z_0$ is defined as

$$\text{Res }(f, z_0) = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

If z_0 is a regular point, then Res $(f(z_0)) = 0$. The residue theorem follows from this definition and the generalized Cauchy integral formula which allows us to calculate a function on a contour enclosing multiple isolated singularities as the sum of the integrals of the function around each isolated singularity.

The residue theorem makes it a lot easier to calculate many contour integrals, since the value of the integral is the same as $2\pi i$ times the sum of the residues of the singularities inside the contour.

6.4 Calculating Residues

Recall from the definition of the Laurent series for a function $f(z)$ about z_0, that

$$b_1 = \frac{1}{2\pi i} \int_C f(w)\, dw.$$

This quantity is called the **residue** of f at z_0. So a guaranteed way to find the residue of a function $f(z)$ at a singularity z_0 is to find the value of b_1 in the Laurent series.

$$\text{Res } f(z_0) = b_1, \tag{6.1}$$

where b_1 is the coefficient of $(z - z_0)^{-1}$ in the Laurent series.

Recall that Taylor and Laurent series can be multiplied together, so if you're trying to find the residue of a complicated function at a point z_0, it is sometimes easier to write the function as the product of two simpler functions. Then if you can replace those simpler functions by Laurent series expanded about z_0, you can multiply out the first several terms of each series to find the b_1 of the multiplied series.

There are other ways of calculating the residue as well. Recall that if z_0 is a simple pole of $f(z)$, then multiplying $f(z)$ by $z - z_0$, will isolate the b_1 term. If we now take the limit as $z \to z_0$, all the a_n terms go to zero and we are left with b_1. In other words, for a simple pole (i.e. a pole of order $m = 1$),

$$\text{Res }(f, z_0) = \lim_{z \to z_0} \left[f(z)(z - z_0) \right]. \tag{6.2}$$

We can also think of this as

$$\text{Res }(f, z_0) = b_1 = \lim_{z \to z_0} \phi(z) = \phi(z_0),$$

where

$$\phi(z) = (z - z_0) f(z).$$

Similarly, if $f(z)$ has a pole of order m, then its Laurent expansion looks like

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}.$$

Multiplying both sides by $(z - z_0)^m$ gives us

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \cdots + b_m.$$

If we now take $m-1$ derivatives of both sides, on the right we end up with some sum containing $(z-z_0)$ raised to some power plus $(m-1)!\,b_1$. That is, the derivative removes the other b_n terms one at a time. If we now take the limit as $z \to z_0$, then the expression in the sum goes to zero, so we have that for a pole of order m, the residue of $f(z)$ at z_0 is

$$\text{Res}\,(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-z_0)^m\right]. \tag{6.3}$$

We can also think of this as

$$\text{Res}\,(f, z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

where

$$\phi(z) = (z-z_0)^m f(z).$$

Note, the formula above can only be used provided that $\phi^{(m-1)}(z_0) \neq 0$. If it is zero, you can often factor something from the top and bottom of $f(z)$ to make this method work. Otherwise, just use the Laurent expansion to find the residue.

Note, a **removable singularity** is a singularity of order $m = 0$ and its residue is always 0. An **essential singularity** also has a residue, but the only way to find it is to do the Laurent expansion and look at b_1.

Tip

After calculating a residue for a function $f(z)$ at a singular point z_k, double-check your work by calculating the Laurent expansion of the function about the singularity. Then, the residue will be the coefficient on the $(z-z_k)^{-1}$ term. It is important to remember that the expansion must be around the singular point z_k—you can't just take the expansion of the function about zero unless $z_k = 0$.

Example 6.4.1

Calculate the residue of

$$f(z) = \frac{1}{4z+1},$$

at $z = -\frac{1}{4}$.

Since $z = -\frac{1}{4}$ is a simple pole, we can use Eq. (6.2).

$$\begin{aligned}
\text{Res}\, f(z_0) &= \lim_{z\to z_0}\left[f(z)(z-z_0)\right] \\
&= \lim_{z\to -\frac{1}{4}}\left[\frac{1}{4z+1}\cdot(z+\frac{1}{4})\right] \\
&= \lim_{z\to -\frac{1}{4}}\left[\frac{1}{4}\right] \\
&= \frac{1}{4}.
\end{aligned}$$

Example 6.4.2

Calculate the residue of

$$f(z) = \frac{1}{z^3(1-z)},$$

at $z = 0$.

To use Eq. (6.1), we need to find the Laurent expansion of $f(z)$. Since we are finding expanding around $z = 0$, we have that $|z| < 1$ and we can express part of

$f(z)$ as a geometric series (as we did earlier) to get

$$\begin{aligned} f(z) &= \frac{1}{z^3} \cdot \frac{1}{1-z} \\ &= \frac{1}{z^3} \cdot \sum_{k=0}^{\infty} z^k \\ &= \sum_{k=0}^{\infty} z^{k-3} \\ &= z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3 + \cdots \end{aligned}$$

Since the coefficient of $(z-0)^{-1}$ is $b_1 = 1$, we have that

$$\text{Res } f(0) = 1.$$

We could also use Eq. (6.3), by first identifying $z = 0$ as a pole of order 3. Then

$$\begin{aligned} \text{Res } f(z_0 = 0) &= \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{1}{z^3(1-z)} (z-0)^3 \right] \\ &= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{1}{1-z} \right] \\ &= \lim_{z \to 0} \left[\frac{1}{(1-z)^3} \right] \\ &= 1. \end{aligned}$$

6.5 Summary: Singularities and Residues

Singularities

There are three types of isolated singularities: removable singularities, poles, and essential singularities.

There are two ways we can determine if an isolated singularity z_0 is removable:

1. We can write the Laurent series for the function. If all $b_n = 0$, then the singularity is removable.
2. We can take the limit $\lim_{z \to z_0} f(z)$. If it exists, then it is a removable singularity.

If the singularity is removable and $f(z)$ is analytic on $N_R^*(z_0)$, then the singularity can be removed by redefining $f(z)$ as

$$f(z) = \begin{cases} \text{old } f(z), & 0 < |z - z_0| < R \\ a_0, & z = z_0. \end{cases}.$$

The a_0 can be found either by looking at the Laurent series or by taking the limit $\lim_{z \to z_0} f(z) = a_0$.

An isolated singularity is a **pole of order** m if the Laurent series has only finite nonzero b_n terms and the largest one is b_m or if

$$\lim_{z \to z_0} (z - z_0)^m f(z),$$

exists, but

$$\lim_{z \to z_0} (z - z_0)^{m-1} f(z),$$

does not exist.

An isolated singularity is an **essential singularity** if the Laurent series has infinitely many nonzero b_n terms.

In general, to classify singularities using the Laurent series, we can use the following process. For a complex function $f(z)$ expanded around z_0, if z_0 is a singularity, then

- if all b_n are zero, then z_0 is a **removable singularity**.
- if only finitely many b_n are nonzero, then z_0 is a **pole of order** m where m is the largest n such that b_n is nonzero. In particular, if b_1 is nonzero, but all other b_n are zero, then z_0 is a **simple pole**.
- if infinitely many b_n are nonzero, then z_0 is an **essential singularity**.

Residues

If $f(z)$ is analytic inside a region enclosed by the closed path C except at n isolated singularities, $z_1, z_2, \ldots, z_n$ then the **residue theorem** states that

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res } (f, z_k).$$

That is, the integral of a function along a closed path C is $2\pi i$ times the sum of the residues of the singularities enclosed by C. When using this theorem, we don't care about any singularities that may appear outside of C.

For a function $f(z)$ with an isolated singularity at z_0,

1. if z_0 is a **removable singularity**, then
$$\text{Res } (f, z_0) = 0.$$
2. if z_0 is a pole of order m, then
$$\begin{aligned} \text{Res } (f, z_0) &= b_1 \\ &= \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, \end{aligned}$$
where $\phi(z) = (z - z_0)^m f(z)$ *and* $\phi(z_0) \neq 0$.
3. if z_0 is an **essential singularity**, then
$$\text{Res } (f, z_0) = b_1.$$

Notice that in all three cases, the residue is equal to the b_1 coefficient of the Laurent series of $f(z)$ expanded about z_0.

Chapter 7

Definite Integration with Residues

7.1 Type 1 Integrals

Consider an integral of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta,$$

where R is a rational function of $\cos\theta$ and $\sin\theta$. If we make the substitution $z = e^{i\theta}$, then as θ runs from 0 to 2π, z runs counterclockwise along the unit circle $|z| = 1$. Then from $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$, we get the substitutions

$$\begin{aligned} \cos\theta &= \frac{1}{2}(z + z^{-1}) \\ \sin\theta &= \frac{1}{2i}(z - z^{-1} \\ d\theta &= \frac{1}{iz}dz. \end{aligned}$$

Now, we can rewrite our integral as a contour integral

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} \tilde{R}(z)\, dz,$$

where $\tilde{R}(z)$ is a rational function of z. Since $\tilde{R}$ is a rational function of z, that is, a ratio of polynomials in z, it is analytic in the unit circle except for a finite number of poles. By finding the poles z_k of $\tilde{R}$, we can use the residue theorem to evaluate the contour integral as $2\pi i$ times the sum of the residues

$$\oint_{|z|=1} \tilde{R}(z)\, dz = 2\pi i \sum \operatorname{Res}\left(\tilde{R}, z_k\right).$$

This method works provided that the denominator of $R(\cos\theta, \sin\theta)$ is never zero for any θ. If $R(\cos\theta, \sin\theta)$ is an even function, then this method also works to evaluate integrals from 0 to π since for even functions, the value of the integral from 0 to π is half the value of the integral from 0 to 2π.

Example 7.1.1

Evaluate

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta.$$

If we let $z = e^{i\theta}$, then our substitutions are

$$\begin{aligned} \cos\theta &= \frac{1}{2}(z + z^{-1}) \\ \cos 3\theta &= \frac{1}{2}(z^3 + z^{-3}) \\ d\theta &= \frac{1}{iz}dz, \end{aligned}$$

so our integral becomes the contour integral

$$\begin{aligned} \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta &= \oint_{|z|=1} \frac{\frac{1}{2}(z^3 + z^{-3})}{5 - 4\left[\frac{1}{2}(z + z^{-1})\right]} \frac{1}{iz}dz \\ &= \frac{1}{2i}\oint_{|z|=1} \frac{z^3 + z^{-3}}{5z - 2z^2 - 2}dz \\ &= -\frac{1}{2i}\oint_{|z|=1} \frac{z^6 + 1}{2z^5 - 5z^4 + 2z^3}dz. \end{aligned}$$

The denominator can be factored as $2z^3(z-2)(z-\frac{1}{2})$, so the integrand as a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. According to the residue theorem, the integral is $2\pi i$ times the sum of the residues of the singularities *inside* the contour. Since $z = 2$ is outside the contour, we only calculate the other two residues, which are easily calculated as

$$\begin{aligned} \text{Res}\left(\tilde{R}, 0\right) &= \frac{21}{8} \\ \text{Res}\left(\tilde{R}, \frac{1}{2}\right) &= \end{aligned}$$

7.2 Type 2 Integrals

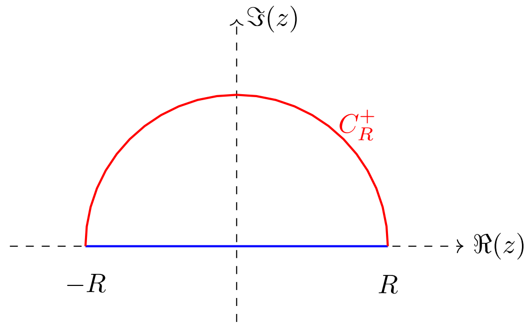
Consider an integral of the form

$$\int_{-\infty}^{\infty} f(x)\,dx.$$

Instead of the real function $f(x)$, consider the complex function $f(z)$. If $f(z)$ is analytic on the upper half of the complex plane except at finite isolated singularities, $z_1, z_2, \ldots, z_n$, then by the residue theorem

$$\int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k),$$

where C_R is the closed contour running from $-R$ to R on the real axis and then over the upper semi-circle of radius R. Note that we want R large enough that our contour encloses all singularities in the upper half plane.



We can split this integral as

$$\int_{C_R} f(z)\,dz = \int_R f(z)\,dz + \int_{C_R^+} f(z)\,dz.$$

where the region R is the real axis taken from $-R$ to R, and C_R^+ is the upper semicircle of radius R taken in the counterclockwise direction.

If we now parametrize the integral along the real part of the axis as $z = x$ where $-R \le x \le R$, then $dz = dx$, and we get

$$\int_{C_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz.$$

Putting together what we have so far, we get

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k) - \int_{C_R^+} f(z)\,dz.$$

If we take R to infinity, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k) - \lim_{R\to\infty} \int_{C_R^+} f(z)\,dz.$$

Now, *if* you can show that

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0,$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k).$$

To show that the limit above is zero, we have to show that the integrand is bounded (using the techniques from a previous chapter), and then show that that bound goes to zero as R goes to infinity.

Be aware that this method has to be modified if $f(z)$ has real zeros. That is, this method presumes that there isn't a pole directly on the x-axis.

Example 7.2.1

Calculate

$$\int_0^{\infty} \frac{1}{(x^2+1)^2}\,dx,$$

using complex methods.

We start by letting

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}.$$

Notice that this function has singularities at $z = \pm i$, so by the residue theorem

$$\int_C \frac{1}{(z^2+1)^2} dz = 2\pi i \operatorname{Res}(f, i),$$

where C is the closed curve containing the real axis on the interval $[-R, R]$ and the upper semicircle of radius $R > 1$. We want $R > 1$ so that the singularity $z = i$ is inside our contour and stays inside as we take R to infinity. We don't care about the singularity at $z = -i$ since it is outside our closed contour. Notice that we can split this integral as

$$\int_C \frac{1}{(z^2+1)^2} dz = \int_{\delta_R} \frac{1}{(z^2+1)^2} dz + \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz,$$

wehre δ_R is the integral along the real axis from $-R$ to R and γ_R is the integral along the upper semicircle of radius R.

By paramtrizing $z = x$, then $dz = dx$ and we let x run from $-R$ to R, so the integral along the real part of the x-axis becomes

$$\int_{\delta_R} \frac{1}{(z^2+1)^2} dz = \int_{-R}^{R} \frac{1}{(x^2+1)^2}\, dx.$$

The integrand is an even function, so

$$\int_{-R}^{R} \frac{1}{(x^2+1)^2}\, dx = 2\int_0^R \frac{1}{(x^2+1)^2}\, dx.$$

Putting together what we have so far, we get

$$2\pi i \operatorname{Res}(f, i) = 2\int_0^R \frac{1}{(x^2+1)^2}\, dx + \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz.$$

Rearranging and letting R go to infinity gives us

$$\begin{aligned}\int_0^\infty \frac{1}{(x^2+1)^2}\, dx \quad &= \quad \pi i \operatorname{Res}(f, i) \\ &\quad -\frac{1}{2} \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz.\end{aligned}$$

Next, we want to show that the integral on the right goes to 0 as R goes to infinity. To do that, we bound the integrand, using the techniques from an earlier chapter. We find that

$$\left| \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz \right| \le \frac{\pi R}{(R^2-1)^2} \sim \frac{1}{R^3},$$

which implies that

$$\frac{1}{2} \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz = 0.$$

Therefore,

$$\int_0^\infty \frac{1}{\quad} dx = \pi i \operatorname{Res}(f, i)$$

Example 7.2.2

Evaluate

$$I = \int_0^\infty \frac{\cos x}{1+x^4}\, dx.$$

Since

$$f(x) = \frac{\cos x}{1+x^4},$$

is an even function, we know that

$$\int_0^\infty \frac{\cos x}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^4}\,dx.$$

We consider now the complex function

$$f(z) = \frac{e^{iz}}{1+z^4}.$$

We know that $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}||e^{-y}| = |e^{-y}|$. Since we are only interested in the upper half of the complex plane, and $y \geq 0$ there, we have that $|e^{-y}| \leq 1$. So we have that

$$\left|\frac{e^{iz}}{1+z^4}\right| \leq \frac{1}{|z|^4}.$$

So by the proof given above, we know that

$$\lim_{R\to\infty}\int_{C_R^+} \frac{e^{iz}}{1+z^4}\,dz = 0,$$

and so

$$\int_{-\infty}^\infty \frac{e^{ix}}{1+x^4}\,dx = 2\pi i \sum_k \text{Res}\left(\frac{e^{iz}}{1+z^4}, z_k\right).$$

Taking the real part of both sides gives us

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^4}\,dx = \Re\left[2\pi i \sum_k \text{Res}\left(\frac{e^{iz}}{1+z^4}, z_k\right)\right],$$

so we only need to find the residues of the singularities of $f(z)$ in the upper half of the complex plane, add them up, and multiply them by $2\pi i$ to get the value of this integral. To obtain the value of the original integral, we have to divide by 2 in order to get the value from 0 to ∞ instead of from $-\infty$ to ∞.

$$I = \int_0^\infty \frac{\cos x}{1+x^4}\,dx = \Re\left[\pi i \sum_k \text{Res}\left(\frac{e^{iz}}{1+z^4}, z_k\right)\right],$$

The denominator $1+z^4$ has four zeros

$$\begin{aligned}
z_1 &= \frac{\sqrt{2}}{2}(1+i) \\
z_2 &= \frac{\sqrt{2}}{2}(-1+i) \\
z_3 &= \frac{\sqrt{2}}{2}(-1-i) \\
z_4 &= \frac{\sqrt{2}}{2}(1-i),
\end{aligned}$$

which are the simple poles of $f(z)$. However, only the first two are in the upper half of the complex plane, so we only have to calculate those residues. Calculating the residue of a general simple pole z_k, we get

$$\begin{aligned}\text{Res }\ (f(z), z_k) &= \lim_{z\to z_k}\left[\frac{e^{iz}}{1+z^4}(z-z_k)\right] = \frac{0}{0} \\ &= \lim_{z\to z_k}\left[\frac{ie^{iz}(z-z_k)+e^{iz}}{4z^3}\right] \\ &= \frac{e^{iz_k}}{4z_k^3} = -\frac{1}{4}z_k e^{iz_k}.\end{aligned}$$

We make use of the fact that $1+z_k^4 = 0$ to realize that the first limit is $\frac{0}{0}$. Then we apply L'Hopital's rule.

So

$$\begin{aligned} I &= \Re\left[\pi i\left(-\frac{1}{4}z_1 e^{iz_1} - \frac{1}{4}z_2 e^{iz_2}\right)\right] \\ &= -\frac{1}{4}\pi\Re\left[iz_1 e^{iz_1} + iz_2 e^{iz_2}\right] \\ &= -\frac{\sqrt{2}}{8}\pi\Re\left[(i-1)e^{i\frac{\sqrt{2}}{2}(1+i)} - (1+i)e^{i\frac{\sqrt{2}}{2}(-1+i)}\right] \\ &= -\frac{\sqrt{2}}{8}\pi e^{-\frac{\sqrt{2}}{2}}\Re\left[(i-1)e^{i\frac{\sqrt{2}}{2}} - (1+i)e^{-i\frac{\sqrt{2}}{2}}\right] \\ &= -\frac{\sqrt{2}}{8}\pi e^{-\frac{\sqrt{2}}{2}}\Re\left[ie^{i\frac{\sqrt{2}}{2}} - e^{i\frac{\sqrt{2}}{2}} - e^{-i\frac{\sqrt{2}}{2}} - ie^{-i\frac{\sqrt{2}}{2}}\right] \\ &= -\frac{\sqrt{2}}{8}\pi e^{-\frac{\sqrt{2}}{2}}\left[-2\sin\frac{\sqrt{2}}{2} - 2\cos\frac{\sqrt{2}}{2}\right] \\ &= \frac{\sqrt{2}}{4}\pi e^{-\frac{\sqrt{2}}{2}}\left[\sin\frac{\sqrt{2}}{2} + \cos\frac{\sqrt{2}}{2}\right] \\ &\approx 0.772138.\end{aligned}$$

7.3 Summary: Integration with Residues

For integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

instead of the real function $f(x)$, consider the complex function $f(z)$. If $f(z)$ is analytic on the upper half of the complex plane except at finite isolated singularities, $z_1, z_2, \ldots, z_n$, then by the residue theorem

$$\int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k),$$

where C_R is the closed contour running from $-R$ to R on the real axis and then over the upper semi-circle of radius R. Note that we want R large enough that our contour encloses all singularities in the upper half plane.

We can split this integral as

$$\int_{C_R} f(z)\,dz = \int_{R} f(z)\,dz + \int_{C_R^+} f(z)\,dz.$$

where the region R is the real axis taken from $-R$ to R, and C_R^+ is the upper semicircle of radius R taken in the counterclockwise direction.

If we now parametrize the integral along the real part of the axis as $z = x$ where $-R \leq x \leq R$, then $dz = dx$, and we get

$$\int_{C_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz.$$

Putting together what we have so far, we get

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k) - \int_{C_R^+} f(z)\,dz.$$

If we take R to infinity, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k) - \lim_{R\to\infty} \int_{C_R^+} f(z)\,dz.$$

Now, *if* you can show that

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0,$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

To show that the limit above is zero, we have to show that the integrand is bounded (using the techniques from a previous chapter), and then show that that bound goes to zero as R goes to infinity.

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