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Last Updated: December 20, 2017

Chapter 1

Real Analysis

1.1 Preliminaries

1.1.1 Number Sets

Natural Numbers

The natural numbers are the set of counting numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Some authors include 0 in the set of natural numbers.

The binary operations that we can perform on natural numbers such that the result is also a natural number include addition and multiplication. We cannot do division, because, e.g., $3/2$ is not a natural number. We cannot do subtraction, because, e.g. $2 - 3$ is not a natural number.

Another property of the natural numbers is that they are ordered. For example, we can say that $1 < 4$.

Integers

The integers are the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Now in addition to addition and multiplication, we can also do subtraction. We say that the integers form a *ring*. We still cannot divide, however.

Rational Numbers

The rational numbers are the set $\mathbb{Q}$ of fractions $\frac{m}{n}$ such that $m, n \in \mathbb{Z}$ and $n \neq 0$. With the rational numbers, our binary operations now include addition, subtraction, multiplication, and division. The rational numbers are said to form a *field*. Like the other number sets we've looked at, the rational numbers are also ordered. Note that the fraction $\frac{m}{n}$ can be thought of simply as an ordered pair (m, n).

What about the identification of unsimplified fractions? A rational number is an equivalence class of ordered pairs under the equivalence relation

$$\frac{m_1}{n_1} = \frac{m_2}{n_2}, \quad \text{if} \quad m_1 n_2 = n_1 m_2.$$

For example, the fractions $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent since $3 \cdot 4 = 2 \cdot 6$.

The rational numbers are not sufficient for us to do calculus. This is because it has gaps. For example, $\sqrt{2}$ is not a rational number.

Prime Numbers

An integer p is a prime number if its only divisors are 1 and p.

FACT: If a and b are integers and p is a prime and p divides ab, then p divides a or p divides b.

Consider $a = 3$ and $b = 4$ then $ab = 12$. Next, consider $p = 2$. It is a prime number, and it divides 12 evenly, therefore, by the fact above, $p = 2$ either evenly divides $a = 3$ or $b = 4$. Remember that p must be a prime. We know that 6 also evenly divides 12, but since 6 is not a prime number, the fact given above does not imply that 6 evenly divides 3 or it evenly divides 4.

Using the fact given above, we will now prove by contradiction that $\sqrt{2}$ is irrational. The claim is that $\sqrt{2}$ is irrational, so we will show that assuming $\sqrt{2}$ is rational leads to a contradiction.

Suppose that there is a rational number $\frac{m}{n} = \sqrt{2}$ where $m, n \in \mathbb{Q}$ and m and n have no common factors. Squaring both sides gives us

$$\left(\frac{m}{n}\right)^2 = 2,$$

which implies that

$$m^2 = 2n^2.$$

Since 2 is prime, by our *fact*, this means that 2 divides $m \cdot m$. In other words, $m = 2 \cdot k$ for some integer k. Therefore,

$$\begin{aligned} m^2 &= 2n^2 \\ (2k)^2 &= 2n^2 \\ 4k^2 &= 2n^2 \\ 2k^2 &= n^2. \end{aligned}$$

But now we have an equation of the same form, and by our fact, this implies that 2 divides $n \cdot n$. Therefore, 2 is a common factor for both m and n. But this contradicts our original assumption (that they have no common factors), so our claim that $\sqrt{2}$ is rational must be false.

Real Numbers

Since $\sqrt{2}$ is not a rational number, our set of rational numbers has gaps. These gaps mean we cannot solve equations like $x^2 - 2 = 0$. Consider the sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

The sequence "wants" to converge, but if you only know about the rational numbers, there is no number for this sequence to converge to. I.e. it does not have a rational limit. In order to do calculus, we need a larger set of numbers that fills in the gaps in the rational numbers. This larger set is the real numbers $\mathbb{R}$.

1.1.2 Sets

A set is a collection of mathematical objects. For example, the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ forms a set. Here, we defined the set by listing its elements. We could also define a set just using words. For example, let A be the set of all integers that are divisible by two. Another common way to define a set is as in the example $B = \{x \in \mathbb{Q} : x^2 < 4\}$. This is read as "$B$ is the set of all rational numbers whose square is less than 4".

One important set is the **null set** or **empty set**

$$\emptyset = \{\}.$$

We can actually construct the natural numbers starting with only the empty set. We define them as follows:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{\emptyset\} \\ 2 &= \{\emptyset, \{\emptyset\}\} \\ 3 &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \end{aligned}$$

We've defined 0 as the empty set, and we've defined 1 as the set containing 0. We define 2 as the set containing 0 and 1, and so on we go.

Next we look at set operations. The **union** of sets A and B written

$$A \cup B,$$

is the set of all elements that are in either A or B or in both. The **intersection**

$$A \cap B,$$

is the set of only those elements that are in both A and B.

The **complement**

$$B \backslash A,$$

is the set of elements that are in B but not in A. Think of it as "B minus A". If B is an *understood* set, then we may write the complement as

$$A^c,$$

which is simply the set of all elements that are not in A. For example, if B is the set of all real numbers, then A^c is understood to be the set of all real numbers that are not in A.

De Morgan's laws are a pair of inference rules that applies to sets. They are

$$\boxed{\begin{aligned} (A \cup B)^c = A^c \cap B^c \\ (A \cap B)^c = A^c \cup B^c. \end{aligned}}$$

Tip:

For De Morgan's laws, think of it as the complement symbol distributing through and flipping the middle symbol.

If we have two sets A and B, then the set inclusion

$$A \subseteq B,$$

means that for every $x \in A$, we also have $x \in B$. In other words, A is contained in B. This is an example of a *partial ordering*. With numbers, we can have *total ordering* as in $x \leq y$. We can also reverse this as

$$B \supseteq A,$$

which means the same thing—set B contains set A. In the special case when the two sets are the same, we write $A = B$, which is equivalent to $A \subseteq B$ and $B \subseteq A$. In other words, A is contained in B and B is contained in A.

As an example of what we can do with set inclusion, consider the infinite sets

$$\begin{aligned} A_1 &= \mathbb{N} = \{1, 2, 3, \ldots\} \\ A_2 &= \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\} \\ A_3 &= \mathbb{N}\backslash\{1, 2\} = \{3, 4, 5, \ldots\} \\ &\vdots \\ A_n &= \{m \in \mathbb{N} : m \geq n\}. \end{aligned}$$

Notice that this is a nested sequence of sets in that A_1 contains A_2, which contains A_3, and so on

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots.$$

Notice that $A_1 \cup A_2 = A_1$ since A_2 doesn't bring anything that is not already in A_1. Similarly, $A_2 \cup A_3 = A_2$, and so on. This means that the union of all the infinite sets is equal to A_1. We can write this as

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots = A_1.$$

What about intersections? The intersection $A_1 \cap A_2$ is just the set A_2. The only elements in A_1 that are not in A_2 are $\{1\}$. Similarly, $A_2 \cap A_3 = A_3$, and so on. If we take the intersection of all infinite sets, we get the null set

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset.$$

1.1.3 Logical Operators

Logical operators operate on mathematical statements to make new mathematical statements. Example mathematical statements include "2 is prime" and "4 is prime". The first statement is true, and the second is false.

Suppose we have the statements P and Q. Then

$$P \wedge Q,$$

means P *and* Q. That is, $\wedge$ is the *and* operator. For $P \wedge Q$ to be true, both P and Q must be true.

The symbol for the *or* operator is $\vee$, and

$$P \vee Q,$$

means P *or* Q. If either P, Q, or both are true then $P \vee Q$ is true.

Tip:

The union of two sets is like the logical 'or' operator. In fact, the two look a little similar with $\cup$ for union and $\vee$ for the logical 'or'. Similarly, the intersection operator $\cap$ is analogous to the logical 'and' operator $\wedge$.

Next, we have the *implication* operator $\implies$. The statement

$$P \implies Q,$$

means P *implies* Q, and it is equivalent to *if* P *then* Q.

The *iff*, which stands for "if and only if" operator has the symbol $\Leftrightarrow$. The statement

$$P \Leftrightarrow Q,$$

means P *is equivalent to* Q. It is the same as saying that $(P \implies Q) \wedge (Q \implies P)$.

To really show the meaning of the different operators, we can use a truth table as shown below.

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T	T
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	F	F	T	T	T

Notice that if P is true, then $P \Rightarrow Q$ is true only if Q is also true. On the other hard, if P is false, then $P \Rightarrow Q$ is true regardless of the value of Q. Similarly, if Q is true, then $P \Leftarrow Q$ is true only if P is also true. On the other hand, if Q is false, then $P \Leftarrow Q$ is true regardless of the value of P. Finally, $P \Leftrightarrow Q$ is true only if P and Q are both true or both false.

The final logical operator that we'll look at is the *not* operator $\neg$. The statement

$$\neg P,$$

means "not P". If P is true, then $\neg P$ is false.

Consider the first of De Morgan's laws $(A \cup B)^c = A^c \cap B^c$. On the left we have the elements that not in A or in B. We can write this using logical notation as $\neg(P \vee Q)$ if we define P as $x \in A$ and Q as $x \in B$. This is the same as saying the elements that are not in A *and* not in B. Logically, we write this as $(\neg P) \wedge (\neg Q)$. So the first of De Morgan's laws in logical notation is $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$. We can write the second law in a similar manner.

In logical notation, **De Morgan's laws** are

$$\boxed{\begin{aligned} \neg(P \vee Q) = (\neg P) \wedge (\neg Q) \\ \neg(P \wedge Q) = (\neg P) \vee (\neg Q). \end{aligned}}$$

A truth table showing the relations for the first of De Morgan's laws is given below.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Our goal is often to prove $P \Rightarrow Q$ for some statements P and Q. **Proof by contrapositive** is one method. The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. It is formed by negating both statements and changing the direction of implication. The original statement and its contrapositive are equivalent. That is

$$\boxed{(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).}$$

To show this, we only need to consider the following truth table.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

1.2 The Real Numbers

1.2.1 Functions

For the sets A and B, a function $f : A \to B$ is a rule or mapping that takes each $x \in A$ and associates it with a single element of B. Then $f(x)$ is the element of B associated with x. A is the **domain** of f, and B is the **codomain** of the function. The codomain is the not the same as the **range**. The range is the set of all numbers that f maps to, but the codomain can be larger.

For example, we can say that $f(x) = x^2$ is a function that maps every $x \in \mathbb{R}$ to a single element in $\mathbb{R}$. Here, the domain is $\mathbb{R}$ and the codomain is $\mathbb{R}$. The actual range however is $[0, \infty)$ which is only half of the codomain.

We can also construct strange functions such as the **Dirichlet function** which returns 1 if x is a rational number and 0 if x is not a rational number

$$I_Q(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}.$$

Note that $I_Q : \mathbb{R} \to \mathbb{R}$, but the range is just the set of two numbers $\{0, 1\}$.

The **absolute value function** is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} = \sqrt{x^2}.$$

The absolute value function is an important one because it gives us a way to measure distance between numbers. The distance from x to y on the real number line is $|x-y|$.

Two important properties of the absolute value function are

$$\begin{aligned} |xy| &= |x||y| \\ |x+y| &\leq |x| + |y|. \end{aligned}$$

The second property is called the **triangle inequality**. These properties can easily be verified by checking the four different cases in which one or both is a negative number.

Note that

$$|x - z| = |(x - y) + (y - z)|,$$

where we've added and subtracted y on the right. By the triangle inequality, we have

$$|x - z| \leq |x - y| + |y - z|.$$

Think of a Cartesian plane with the three points x, y, and z. On the left we have $|x - z|$ which is simply the distance between x and z. On the right, we have the sum of the distance between x and y and the distance between y and z. So if x, y, and z are the vertices of a triangle, this equation says that the length of one side is less than or equal to the sum of the lengths of the other two sides. Hence the name "triangle inequality".

1.2.2 Proofs

Consider the following theorem.

Theorem:

Two real numbers a and b are equal iff for every real $\varepsilon > 0$, we have $|a - b| < \varepsilon$.

First, we have to understand what's being claimed. We have two mathematical statements:

- P: Two real numbers a and b are equal.
- Q: For every real $\varepsilon > 0$, we have $|a - b| < \varepsilon$.

We want to show that $P \Leftrightarrow Q$. That is, we need to show that

1. $P \implies Q$
2. $Q \implies P$

Now, we just treat these two separately.

For the first one, we suppose that a and b are equal. Then $a - b = 0$, so $|a - b| = 0 < \varepsilon$ for every $\varepsilon > 0$. This takes care of the $P \implies Q$ part.

To prove $Q \implies P$, we'll use a proof by contrapositive. That is, we show that $\neg P \implies \neg Q$. Now we have to interpret what $\neg P$ and $\neg Q$ mean given our definition of P and Q. Note that "not(for all)" means the same as "there exists...". So "not(Q)" means "there exists an $\varepsilon > 0$ such that $|a - b| \geq \varepsilon$". For the second statement, "not(P)" means simply "$a \neq b$". In other words, we have to show that $a \neq b$ implies that there exists an $\varepsilon > 0$ such that $|a - b| \geq \varepsilon$.

Suppose $a \neq b$, then $a - b \neq 0$, so $|a - b| > 0$. Thus taking $\varepsilon = |a - b|$, we have $\varepsilon > 0$ and $|a - b| \geq \varepsilon$. This completes our proof of the theorem.

1.2.3 Induction

Induction is a proof method used when you have infinitely many mathematical statements $S_1, S_2, S_3, \ldots$ such that

1. S_1 is true
2. For every n, if S_n is true, then S_{n+1} is also true. That is, for all n, $S_n \implies S_{n+1}$.

If these conditions are true, then for every n, S_n is true.

Example:

Use induction to show that the sequence

$$x_{n+1} = \frac{1}{2}x_n + 1, \quad x_1 = 1$$

is increasing. That is, show that each term is greater than or equal to the one that came before it.

The first several elements of this sequence are

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots .$$

To start, we let S_n be the statement that $x_{n+1} \geq x_n$. We want to prove that S_n is true for every n. To use induction, we have to check the two conditions

1. $S_1 =$"$\frac{3}{2} \geq 1$"
2. For every n, $S_n \implies S_{n+1}$

The first is obviously true since $\frac{3}{2} > 1$. For the second statement, we want to show that if $x_{n+1} \geq x_n$, then $x_{n+2} \geq x_{n+1}$. To do so, we multiply both sides by $\frac{1}{2}$ and add 1.

$$\begin{aligned} x_{n+1} &\geq x_n \\ \frac{1}{2}x_{n+1} &\geq \frac{1}{2}x_n \\ \frac{1}{2}x_{n+1} + 1 &\geq \frac{1}{2}x_n + 1 \\ x_{n+2} &\geq x_{n+1}. \end{aligned}$$

We've now shown that if $x_{n+1} \geq x_n$, then $x_{n+2} \geq x_{n+1}$, completing the proof by induction.

We could also take a slightly different approach by replacing x_{n+1} by its definition and then isolating x_n.

$$\begin{aligned} x_{n+1} &\geq x_n \\ \frac{1}{2}x_n + 1 &\geq x_n \\ 1 &\geq \frac{1}{2}x_n \\ 2 &\geq x_n. \end{aligned}$$

Now, if $x_n \leq 2$, then $x_{n+1} = \frac{1}{2}x_n + 1 \leq \frac{2}{2} + 1 = 2$. Therefore, $x_n \leq 2$ implies that $x_{n+1} \leq 2$ and the proof is complete.

Incidentally, this second approach also shows that 2 is an upper bound of the sequence.

1.2.4 The Axiom of Completeness

Previously, we showed that the rational numbers form an ordered field. It is ordered because if $a < b$ and $b < c$, then $a < c$. Furthermore, if $a < c$, then $a + d < c + d$. If $a < c$, and $d > 0$, then $ad < cd$. It is a field because multiplication and its regular properties hold. We also showed that the real numbers form an ordered field. The one further property of $\mathbb{R}$ is that it satisfies the *axiom of completeness*, which we will define soon.

Definition: Upper bound

Suppose $A \subseteq \mathbb{R}$, and $b \in \mathbb{R}$, then b is an **upper bound** for A if for every $a \in A$, $b \geq a$.

In other words, if we have a subset A of the real numbers, then a real number b is an upper bound for A if b is larger than every element of A. We say that A is "bounded above" if there is a $b \in \mathbb{R}$ such that b is an upper bound for A.

Consider the set $A = \{1, 2, 3\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$. The numbers $b_1 = 11$ and $b_2 = 3$ are both upper bounds for A. The set $\mathbb{N}$ is not bounded above. We cannot find any number that is larger than all elements of $\mathbb{N}$. Now consider the negatives of the natural numbers $M = \{-1, -2, -3, \ldots\}$. This set is now bounded above because, for example, $b = -1$ is greater than or equal to every element in M.

Note, the **extended real numbers** are the real numbers plus positive and negative infinity. That is, they include infinity as a number. We are not using the extended real numbers in $\mathbb{R}$ because they cause problems.

Definition: Supremum

The real number $s \in \mathbb{R}$ is the **least upper bound** or **supremum** of a set $A \subseteq \mathbb{R}$ if

1. s is an upper bound for A, and
2. if b is an upper bound for A, then $b \geq s$.

In the second requirement, we are just saying that s is smaller than (or equal to) any other upper bound.

For the set $A = \{1, 2, 3\}$ and $M = \{-1, -2, -3, \ldots\}$, the least upper bound is just the maximum of the set. We write it as $\sup A = 3$ and $\sup M = -1$.

We've defined upper bounds and least upper bounds. There are anologous definitions for lower bounds and greatest lower bounds.

Definition: Infinum

The real number $s \in \mathbb{R}$ is the **greatest lower bound** or **infinum** of a set $A \subseteq \mathbb{R}$ if

1. s is a lower bound for A, and
2. if b is a lower bound for A, then $b \leq s$.

Example:

Find the supremum and infinum of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

This is the set that starts

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

Clearly, the supremum is sup $A = 1$. It also seems clear that inf $A = 0$, but to prove this, we need to show that the two requirements are satisfied.

The first requirement is to show that 0 is *a* lower bound for A. To do that we simply set it less than or equal to $\frac{1}{n}$ then multiply across by n

$$\begin{aligned} 0 &\leq \frac{1}{n} \\ 0 &\leq 1. \end{aligned}$$

Since the second relation is clearly true, the first must also be.

The second requirement is to show that every other lower bound for A is smaller than 0, showing that 0 is the greatest lower bound. In other words, we want to show that the statement "b is a lower bound" implies that $b \leq 0$.

To show this, we try a proof by contrapositive. We'll show that

$$\text{not}(b \leq 0) \implies b \text{ is not a lower bound.}$$

This translates to

$$b > 0 \implies b \text{ is not a lower bound,}$$

which means that if $b > 0$, then you can find a number in A that is less than b.

Suppose $b > 0$. Choose $n \in \mathbb{N}$ with $n > \frac{1}{b}$, then $b > \frac{1}{n}$ is an element in A. Therefore, b is not a lower bound for A.

Notice that inf $A = 0$ is not an element of A.

How would we prove that

$$-\inf(A) = \sup(-A),$$

for a set $A \subseteq \mathbb{R}$?

Tip:

The key to proving things is to have a good understanding of the definitions.

Take another look at the definition of the infinum. To prove that a number is the infinum of a set, we have to prove two statements.

We'll start by letting $s = \sup(-A)$. We need to show that $-s = \inf(A)$. Since s is *an* upper bound for $-A$, we know that $s \geq -a$ for every $a \in A$. Therefore, $-s \leq a$ for every $a \in A$. This tells us that $-s$ is *a* lower bound for A, proving the first statement in the definition of the infinim.

Now, suppose t is a lower bound for A, then for every $a \in A$, $t \leq a$ implies that $-t \geq -a$. Hence, $-t$ is an upper bound for the set $-A$. Since s is the *least* upper bound for $-A$, and $-t$ is *an* upper bound for $-A$, it follows that $s \leq -t$. This implies $t \leq -s$, therefore, $-s$ is the *greatest* lower bound for A, proving the second statement in the definition of the infinum.

Theorem: Axiom of Completeness

Every non-empty set of real numbers which is bounded above has a least upper bound. This least upper bound is also a real number.

Now that we have the axiom of completeness, we can define the real numbers $\mathbb{R}$ as an ordered field that satisfies the axiom of completeness.

We could also define the axiom of completeness in terms of greatest lower bound. However, these two versions are equivalent, so we don't need both.

Note that the rational numbers do not satisfy the axiom of completeness. Suppose you were to consider the set of all rational numbers whose squares are less than 5. This means the set is bounded above. For example, 3 is an upper bound because $3^2 > 9$. What is the least upper bound? The condition $r^2 < 5$ is the same as the condition $r < \sqrt{5}$. The least upper bound of our set is $\sqrt{5}$, and it is clearly not a rational number. The rational numbers do not satisfy the axiom of completeness, but the real numbers do. In that sense, the axiom of completeness is what distinguishes $\mathbb{R}$ from $\mathbb{Q}$.

Lemma:

Assume that s is an upper bound for a nonempty set $A \subseteq \mathbb{R}$, then s is a least upper bound for A if and only if for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$.

Recall that the definition of the supremum of a set A has two criteria. The second of those criteria is:

P: If b is any upper bound for A, then $s \leq b$, where s is the supremum of A.

From our lemma, consider this other statement

Q: For every $\varepsilon > 0$, there is an $a \in A$, such that $a > s - \varepsilon$.

We want to show that P and Q are equivalent.

To show that $P \implies Q$, we suppose that P holds and that $\varepsilon > 0$. Then $s - \varepsilon < s$, so by the contrapositive of P, $s - \varepsilon$ is not an upper bound for A, so there is an $a \in A$ with $a > s - \varepsilon$.

Next, we need to show that $Q \implies P$. Suppose that Q is true and $b < s$. Letting $\varepsilon = s - b > 0$, then from Q, we know that there is an $a \in A$ with $a > s - \varepsilon = b$, so b is not an upper bound for A. We have now proved the lemma.

Theorem: Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that for each n, $I_{n+1} \subseteq I_n$. Then the nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ has a nonempty intersection. That is

$$\bigcap_{n=1}^{\infty} I_n \neq \phi.$$

What this means is that we have a series of intervals (each subsequent one is inside the previous) that is narrowing down on some point on the number line. According to this theorem, the final interval, which has narrowed down to a point, contains a real number. This tells us that there are no gaps in the real numbers. I.e. there is no empty interval on the real number line. We want to prove this theorem using the axiom of completeness.

We are looking for a number x that is greater than all a_n and less than all b_n (since we want this number to be in the "innermost" interval). In other words, we know that x is an upper bound for the set of a_n. Let A be the set of all a_n, which are the left-sides of our intervals. Notice that b_1 is an upper bound for A. We know that A is nonempty (e.g. it contains a_1) and bounded above (by each b_n), so by the axiom of completeness, it has a least upper bound, which we'll call $x = \sup A$.

Since x is an upper bound for A, we have $x \geq a_n$ for all n. For each n, b_n is an upper bound for A, so since x is the least upper bound, we have that $b_n \geq x$. So $x \in I_n$ for each n giving us

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

Theorem: Archimedean Property

1. Given any real number $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ with $n > x$.
2. Given any $0 < y \in \mathbb{R}$, there is an $n \in \mathbb{N}$ with $\frac{1}{n} < y$.

The Archimedean property is just the statement that there is no real number that is larger than all natural numbers. Second, there is no positive real number that is smaller than all inverse natural numbers.

We can prove the first statement of the Archimedean property by contradiction. Suppose that the statement is false, then there is an $x \in \mathbb{R}$ such that $x \geq n$ for every $n \in \mathbb{N}$.

To see that this is indeed the negation of the first statement, it helps to write the first statement formally, and then formally negate it. The phrase "for all x" is symbolized as $\forall_x$. The phrase "there exists an n", is symbolized as $\exists_n$. Finally, "such that $n > x$" is symbolized simply by $n > x$. So the entire statement "for all x there exists an n such that $n > x$ can be written symbolically as

$$\forall_x \, \exists_n : n > x.$$

If we now negate this entire statement, we get

$$\begin{aligned} \text{not}(\forall_x \, \exists_n : n > x) &= \exists_x \ \text{not}(\exists_n : n > x) \\ &= \exists_x \forall_n : \ \text{not}(n > x) \\ &= \exists_x \forall_n : n \leq x. \end{aligned}$$

Notice that in general, $\text{not}(\forall) = \exists$ and $\text{not}(\exists) = \forall$. Also, $\text{not}(n > x) = n \leq x$.

So x is an upper bound for $\mathbb{N}$, and so by the axiom of completeness, there is a least upper bound $y = \sup \mathbb{N}$. Now, $y - \frac{1}{2} < y$, so $y - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. Therefore, there is an $n \in \mathbb{N}$ with $n > y - \frac{1}{2}$. Now we add

1 to both sides to get

$$n + 1 > y + \frac{1}{2} > y.$$

But $n + 1 \in \mathbb{N}$, and y is not the least upper bound for $\mathbb{N}$, contradicting our premise. We've proved by contradiction that the first statement of the Archimedean property is true.

For the second statement in the Archimedean property, we suppose that $0 < y \in \mathbb{R}$. By the first statement of the Archimedean property, we know there is an $n \in \mathbb{N}$ with $n > \frac{1}{y}$, so $y > \frac{1}{n}$.

Theorem: Density of $\mathbb{Q}$ in $\mathbb{R}$

Suppose $a < b$ are real numbers. There is an $r \in \mathbb{Q}$ with $a < r < b$.

In other words, there is a rational number between any pair of real numbers. This theorem is about the density of $\mathbb{Q}$ in $\mathbb{R}$. To prove this, we need $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a < \frac{m}{n} < b$. Since $n > 0$, we can multiply through by n and find an integer m such that $na < m < nb$.

The idea is to make the gap between na and nb greater than 1. That is, we want $nb - na > 1$ or

$$n > \frac{1}{b - a}.$$

We know by the Archimedean property that we can find an $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$, so that $nb - na > 1$. So we now have a gap that is larger than 1.

Now we let m' be the largest integer that is less than na. If we let $m = m' + 1$, that is we add 1 to m', then we have an $m \in \mathbb{Z}$ that is between na and nb.

1.2.5 Cardinality

The cardinality of a set is a measure of the size of a set. For example, the set $\{2, 3, 5\}$ has a cardinality of 3 since it contains three elements. The set of natural numbers has infinite cardinality. The set of integers also has infinite cardinality, though in some sense, it has twice as many elements as the natural numbers. The set of rational and real numbers also have infinite cardinality. Are these infinite cardinalities all the same? First, we have to look at some definitions.

Definition: One-to-one

A function $f : A \to B$ is **one-to-one** if whenever $a_1, a_2 \in A$ with $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$. A one-to-one function may also be called **injective**.

For the graphical definition, a function is one-to-one if it satisfies the horizontal line test. For example, $f(x) = x^2$ is not one to one because for $a_1 = 1$ and $a_2 = -1$, we have $a_1 \neq a_2$, but $f(a_1) = f(a_2)$.

Definition: Onto

A function $f : A \to B$ is **onto** if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. An onto function may also be called **surjective**.

For the graphical definition, a function is onto if it hits every y-value. For example, the function $f(x) = x^2$ is not onto because there exists, for example, $b = -1$ but no a such that $f(a) = -1$. The function $f(x) = x^3$ *is* onto because it hits every y-value. The function $f(x) = \tan^{-1} x$ is one-to-one but not onto.

Definition: Bijective

A function $f : A \to B$ is a **one-to-one correspondence** if it is one-to-one and onto. Such a function may also be called **bijective**.

Definition:

Two sets A and B have the same cardinality, denoted $A \sim B$, if there is a bijective function $f : A \to B$.

Example:

Show that the interval (a, b) has the same cardinality as the real numbers.

By the definition given above, to show that the cardinality of two sets is the same, we need to find a bijective function that takes the elements from one set to the elements of the other set. In our case, we want to find a one-to-one and onto function that maps the elements of (a, b) to $\mathbb{R}$.

To do this, imagine the graph of some function with a domain of at least (a, b) and with a range equal to $\mathbb{R}$. We can imagine a function that approaches $-\infty$ as one approaches a from the right and approaches $+\infty$ as one approaches b from the left. To make sure that the range is all of $\mathbb{R}$, we want the function to have a zero somewhere between a and b. Constructing such a function is relatively easy. An example is

$$f(x) = \frac{2x - b - a}{(-x + a)(x - b)}.$$

For example, the two sets $A = \{1, 8, 3\}$ and $\{6, 1, 3\}$ have the same cardinality because we could define a bijective function like $f(1) = 6$, $f(8) = 1$, and $f(3) = 3$. Both of these sets have the same cardinality as the set of the first three natural numbers $\{1,2,3\}$, so we say they have cardinality 3.

Consider the natural numbers $\mathbb{N} = \{1,2,3,4,5,\ldots\}$ and the integers $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. We can assign each integer to a natural number with a function like

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}.$$

This is a bijective function, so $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

What about the rational numbers? Can we show that $\mathbb{Q}$ has the same cardinality as $\mathbb{N}$? We can think of rational numbers as pairs of natural numbers (n,m). Therefore, we can think of $\mathbb{Q}$ as having the same cardinality as $\mathbb{N} \times \mathbb{N}$. Now we want to show that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

Suppose we create a table of all the pairs of natural numbers. The first row of our table starts like $(1,1),(1,2),(1,3),\ldots$. The second row starts as $(2,1),(2,2),(2,3),\ldots$ and so on. We can now draw diagonal lines connecting each of these pairs of numbers. We start at the top left corner of our table, then we zigzag diagonally up and down the table so that we will end up hitting each pair. We can label them, starting with $1 = (1,1)$, $2 = (2,1)$, $3 = (1,2)$, $4 = (1,3)$, $5 = (2,2)$, and so on. Since we can assign every pair to a unique natural number, our method is a bijective function, and therefore, $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

Definition: Countable

If $A \sim \mathbb{N}$, we say A is *countable*.

Definition: Uncountable

If A is an infinite set, but not $A \sim \mathbb{N}$, then we say that A is *uncountable*.

In other words, if a set has the same cardinality as $\mathbb{N}$, it is called a **countable set**. If it has a larger cardinality, it is called an **uncountable set**. That is, if the set is infinite, but doesn't have the same cardinality as $\mathbb{N}$. The set of real numbers is uncountable.

If a set is countable, its elements are enumerable. A sequence $x_1, x_2, x_3, \ldots$ is a function from $\mathbb{N}$ into some set A. So if A is countable, you can enumerate the set $A = \{x_1, x_2, x_3, \ldots\}$.

Theorem:

The set of real numbers $\mathbb{R}$ is uncountable.

We'll prove this by contradiction. To do so, we'll assume that $\mathbb{R}$ is countable, then there must be a way to order $\mathbb{R}$ into an enumerable set $\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$. To show the contradiction, we'll use the closed intervals requirement of the nested interval property.

Choose $b_1 < x_1$ and $a_1 < b_1$ then our first interval is $I_1 = [a_1, b_1]$ and $x_1 \notin I_1$. Now, we make nested intervals. We choose the second interval as follows:

- If $x_2 \notin I_1$, set $I_2 = I_1$.
- If $x_2 \in I_1$, and $x_2 \neq a_1$, then choose b_2 such that $a_1 < b_2 < x_2$, and $a_2 = a_1$. Then $I_2 = [a_2, b_2] \subseteq I_1$ but $x_2 \notin I_2$.
- If $x_2 \in I_1$ and $x_2 = a_1$, then choose a_2 with $x_2 < a_2 < b_1$ and $b_2 = b_1$. Then $I_2 = [a_2, b_2] \subseteq I_1$, but $x_2 \notin I_2$.

We repeat this process for I_3, I_4, and so on, infinite times. In this way, we create a set of nested intervals such that the innermost interval contains none of our $\{x_1, x_2, x_3, \ldots\}$. However, by the nested interval property, we know that there exists an

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

That is, the innermost interval must contain a real number x. But this means that our set $\{x_1, x_2, x_3, \ldots\}$ cannot contain all the real numbers, which contradicts our initial assumption that $\mathbb{R}$ is enumerable.

Theorem:

If $A \subseteq B$ and B is countable, then A is finite or A is countable.

Theorem:

If $A_1, A_2, \ldots, A_n$ are all countable, then their union $\bigcup_{j=1}^{n} A_j$ is countable.

Theorem:

If $A_1, A_2, A_3, \ldots$ (infinite sets) are all countable, then their union $\bigcup_{j=1}^{n} A_j$ is countable.

1.2.6 Cantor's Theorem

In this section, we briefly discuss Cantor's theorem and power sets without going into much detail. **Cantor's diagonalization argument** is another proof that $\mathbb{R}$ is uncountable.

Given a set A, we write 2^A to mean the set of all subsets of A. This is the idea of **power sets**.

Suppose that $A = \{2, 7, 11\}$, then

$$2^A = \{\{2, 7, 11\}, \{2, 7\}, \{2, 11\}, \{7, 11\}, \{2\}, \{7\}, \{11\}, \{\}\}.$$

Notice that the power set 2^A contains 2^3 elements. In general, if A is finite and has n elements, then its power set 2^A will have 2^n elements.

Interestingly, the power set of the natural numbers is not countable. In fact, the power set of $\mathbb{N}$ has the same cardinality as the real numbers

$$2^{\mathbb{N}} \sim \mathbb{R}.$$

The cardinality of the natural numbers is denoted "aleph-null" and written $\aleph_0$. The cardinality of the real numbers is denoted $\aleph_1$.

We can find even bigger cardinalities. The power set of the real numbers has a larger cardinality than the real numbers. In fact, the power set of the real numbers has the same cardinality as the power set of the power set of the natural numbers

$$2^{\mathbb{R}} \sim 2^{2^{\mathbb{N}}}.$$

This cardinality is denoted $\aleph_2$. We can of course continue, taking power sets of power sets of power sets, and we find that

$$\aleph_0 < \aleph_1 < \aleph_2, \ldots$$

Is there a cardinality between $\aleph_0$ (the cardinality of the natural numbers) and $\aleph_1$ (the cardinality of the real numbers? The **continuum hypothesis** states that there is not. However, we cannot prove nor disprove this hypothesis within the set theory that provides the foundation for modern mathematics. In fact, the foundation of mathematics has been proved to be either incomplete or inconsistent.

1.3 Sequences and Series

1.3.1 Limits of Sequences

Definition: Sequence

A sequence is a function whose domain is $\mathbb{N}$.

For example, $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is a sequence since it can be described as the function $f(n) = \frac{1}{n}$ whose domain is $\mathbb{N}$.

Some different ways to denote sequences include the following:

$$\begin{aligned} \left(\frac{1+n}{n}\right)_{n=1}^{\infty} &= 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots \\ \left\{\frac{1+n}{n}\right\}_{n=1}^{\infty} &= 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots \end{aligned}$$

Keep in mind that a sequence is not the same thing as a set. A sequence is a set plus knowledge of how the elements are ordered.

A sequence can also be defined recursively as in

$$\{a_n\}_{n=1}^{\infty}, \text{ where } a_1 = 2, \text{ and } a_{n+1} = \frac{1+a_n}{2}.$$

This sequence starts

$$2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \ldots$$

Definition: Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers **converges** to a real number a if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that for all $n \geq N$, we have $|a_n - a| < \varepsilon$.

In short, a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a if

$$\lim_{n\to\infty} a_n = a.$$

The statement $|a_n - a| < \varepsilon$ means $a_n < a + \varepsilon$ and $a_n > a - \varepsilon$. It is equivalent to saying that a_n is in the interval $(a - \varepsilon, a + \varepsilon)$.

The interval $(a - \varepsilon, a + \varepsilon)$ or the set of $x \in \mathbb{R}$ whose distance from a is less than ε is sometimes called the **ε-neighborhood** of a and is sometimes denoted $V_\varepsilon(a)$.

Symbolically, we would write the definition of convergence as

$$\forall_{\varepsilon>0} \exists_N \forall_{n\geq N} \ |a_n - a| < \varepsilon.$$

How do we show that a sequence like

$$a_n = \frac{1}{\sqrt{n}},$$

converges? If we consider how the sequence behaves for large n, it becomes intuitively obvious that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

We want to prove this rigorously using our definition of convergence. We want to show that for every $\varepsilon > 0$, there exists some natural number N such that for every term past a_N we have $|a_n - 0| < \varepsilon$.

We start with $\varepsilon > 0$. We want $|a_n - a| < \varepsilon$, so we plug in our values and solve for n.

$$\begin{aligned} \left|\frac{1}{\sqrt{n}} - 0\right| &< \varepsilon \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \sqrt{n} &> \frac{1}{\varepsilon} \\ n &> \frac{1}{\varepsilon^2}. \end{aligned}$$

This tells us to choose $N > \frac{1}{\varepsilon^2}$. Now we are ready to actually write our proof:

"Let $\varepsilon > 0$. Choosing N to be a natural number with $N > \frac{1}{\varepsilon^2}$, then for $n \geq N$, we have $n > \frac{1}{\varepsilon^2}$, so $\sqrt{n} > \frac{1}{\varepsilon}$, so $\frac{1}{\sqrt{n}} < \varepsilon$, so $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$."

Definition: Convergent

A sequence of real numbers $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is **convergent** if there exists an $a \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = a.$$

In other words, that a sequence is convergent is equivalent to the statement

$$\exists_{a\in\mathbb{R}} \forall_{\varepsilon>0} \exists_N \forall_{n\geq N} \ |a_n - n| < \varepsilon.$$

Definition: Divergent

A sequence is **divergent** if it is not convergent.

So that a sequence is divergent, is equivalent to

$$\text{not} \left(\exists_{a\in\mathbb{R}} \forall_{\varepsilon>0} \exists_N \forall_{n\geq N} \ |a_n - n| < \varepsilon\right).$$

This is equivalent to

$$\forall_{a\in\mathbb{R}} \exists_{\varepsilon>0} \forall_N \exists_{n\geq N} \ |a_n - n| \geq \varepsilon.$$

An example of a divergent sequence is the sequence

$$1, -1, 1, -1, 1, -1, \dots$$

To prove that this sequence is divergent, we let $a \in \mathbb{R}$, then take $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. Now there are two cases.

1. If $a \geq 0$, we take n to be an even number (so all $a_n = -1$) such that $n \geq N$, then $|a_n - a| \geq 1$. This is greater than ε, so divergence has been proved for this case.
2. If $a < 0$, we take n to be an odd number (so all $a_n = 1$) such that $n \geq N$, then $|a_n - a| \geq 1$. This is greater than ε, so divergence has been proved for this case too.

Definition: Uniqueness of limits

If $\lim_{n\to\infty} a_n = a$, and $\lim_{n\to\infty} a_n = b$, then $a = b$.

Definition: Boundedness

A sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is **bounded** if there is a real number M such that for all n, $|a_n| \leq M$.

Symbolically, we would write this as

$$\exists_M \forall_n \ |a_n| \leq M.$$

In other words, every term in the sequence is in $[-M, M]$, so there's no blowing up to $\pm\infty$.

Theorem:

Every convergent sequence is bounded.

To prove this theorem, we let $\{a_n\}_{n=1}^{\infty}$ be convergent, so there exists an $a = \lim_{n\to\infty} a_n$. We can choose an arbitrary ε, so choose $\varepsilon = 1$, then find N such that for all $n \geq N$, $|a_n - a| < 1$. Then for all $n \geq N$, $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$. Thus, if $n \geq N$, then every term of the sequence is less than $1 + |a|$. What about the terms of the sequence corresponding to $n < N$. If we want to find the bound M, we have to consider these terms as well. Fortunately, we know that there are only finite terms with $n < N$, so we look at all of them and choose

$$M = \max\left(1 + |a|, |a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|\right),$$

then $M \in \mathbb{R}$, and for every n, we have that $|a_n| < M$.

Be aware that boundedness does not imply convergence. For example, the sequence

$$(-1)^n = \{-1, 1, -1, 1, -1, \dots\},$$

is bounded with $M = 1$, but it is not convergent. However, you can always pull a convergent sequence out of a bounded sequence. In this case, the subsequences consisting of only the odd terms or only the even terms are convergent.

Earlier, we showed an example of proving that

$$\lim_{n\to\infty} a_n = a,$$

for a sequence $\{a_n\}_{n=1}^{\infty}$ by directly solving

$$|a_n - a| < \varepsilon,$$

for n in terms of ε as a first step. When this doesn't work, you have to modify your approach by doing what might be called the *comparison test*.

Suppose you're asked to prove the limit

$$\lim_{n\to\infty} \frac{2n^2}{n^3 + 10} = 0,$$

using the definition of convergence for a limit. We want to end up with

$$\left| \frac{2n^2}{n^3 + 10} - 0 \right| < \varepsilon.$$

However, we cannot easily solve this relation for n. What we do instead, is look at a similar problem. Consider instead the problem

$$\left| \frac{2n^2}{n^3} - 0 \right| < \varepsilon.$$

The reason we do this is because we know that

$$\frac{2n^2}{n^3 + 10} < \frac{2n^2}{n^3},$$

so if we can show that

$$\frac{2n^2}{n^3} < \varepsilon,$$

then it follows that

$$\frac{2n^2}{n^3 + 10} < \varepsilon.$$

Solving the easier problem for n is trivial, and we get $n > \frac{2}{\varepsilon}$.

So our proof would go like this: Let $\varepsilon > 0$. Choose $N > \frac{2}{\varepsilon}$, then for all $n \geq N$, we have that $n > \frac{2}{\varepsilon}$, so $\frac{2}{n} < \varepsilon$, so $\frac{2n^2}{n^3} < \varepsilon$, so $\frac{2n^2}{n^3+10} < \varepsilon$, and therefore,

$$\left| \frac{2n^2}{n^3 + 10} - 0 \right| < \varepsilon.$$

1.3.2 Algebraic Limit Theorem

Theorem: Algebraic Limit Theorem

Suppose

$$\lim_{n\to\infty} a_n = a, \quad \text{and} \quad \lim_{n\to\infty} b_n = b,$$

then

1. If $c \in \mathbb{R}$, then
$$\lim_{n\to\infty} ca_n = ca.$$
2. $$\lim_{n\to\infty} (a_n + b_n) = a + b.$$
3. $$\lim_{n\to\infty} a_n b_n = ab.$$
4. $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}, \quad \text{provided } b \neq 0.$$

We will prove each of the four statements of this theorem.

To prove the first statement, we start with the definition of convergence. The limit $\lim x_n = x$ means that for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$, $|x_n - x| < \varepsilon$.

Let $\varepsilon > 0$, then

$$|ca_n - ca| = |c||a_n - a| < \varepsilon.$$

This gives us

$$|a_n - a| < \frac{\varepsilon}{|c|} = \tilde{\varepsilon}.$$

On the right we've defined the quantity $|a_n - a|$ to be less than a new epsilon. Notice that we have a division by zero problem if $c = 0$. This is not a real problem, if we note that $ca_n = 0, 0, 0, \dots$ converges to zero.

So our proof for the first statement is: If $c = 0$, we are done since $\{0\}_{n=1}^{\infty}$ converges to zero. Otherwise let $\varepsilon > 0$. Then, since $\lim a_n = a$, we can find, by applying the definition of convergence with $\tilde{\varepsilon} = \frac{\varepsilon}{|c|}$, an N such that for all $n \geq N$, $|a_n - a| < \frac{\varepsilon}{|c|}$. Then if $n \geq N$, we have

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$$

This completes our proof.

To prove the second statement, we want

$$|a_n + b_n - (a + b)| < \varepsilon.$$

By the triangle inequality,

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b|.$$

Since we are given that the two separate limits exist, we know that

$$|a_n - a| + |b_n - b| < \varepsilon + \varepsilon = 2\varepsilon.$$

We can also redefine the separate limits such that $|a_n - a| < \frac{\varepsilon}{2}$ and $|b_n - b| < \frac{\varepsilon}{2}$, then we get the combined limit

$$|a_n - a| + |b_n - b| < \varepsilon.$$

Since $\lim a_n = a$, we know that $|a_n - a| < \frac{\varepsilon}{2}$ for any a_n past some $n \geq N_1$. Also, since $\lim b_n = b$, we know that $|b_n - b| < \frac{\varepsilon}{2}$ for any b_n past some $n \geq N_2$. To ensure that we pick an N large enough to satisfy both, we just choose the largest of N_1 and N_2.

We are now ready to write the proof for the second statement: Let $\varepsilon > 0$. Applying the convergence of $\{a_n\}$ with $\tilde{\varepsilon} = \frac{\varepsilon}{2}$, we can find an N_1 such that for all $n \geq N_1$, we have $|a_n - a| < \frac{\varepsilon}{2}$. Similarly, N_2 such that for all $n \geq N_2$, we have $|b_n - b| < \frac{\varepsilon}{2}$. Then we set $N = \max(N_1, N_2)$. Now, if $n \geq N$, we have $n \geq N_1$ and $n \geq N_2$ so

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This concludes the proof.

To prove the third statement, we want

$$|a_n b_n - ab| < \varepsilon.$$

We know how to control $a_n - a$ and $b_n - b$, so we want to separate the a's and b's.

$$\begin{aligned} a_n b_n - ab &= a_n(b_n - b) + a_n b - ab \\ &= a_n(b_n - b) + b(a_n - a). \end{aligned}$$

We want

$$|a_n(b_n - b) + b(a_n - a)| < \varepsilon.$$

By the triangle inequality,

$$|a_n(b_n - b) + b(a_n - a)| < |a_n||b_n - b| + |b||a_n - a|.$$

If

$$\begin{aligned} |a_n||b_n - b| &< \frac{\varepsilon}{2} \\ |b||a_n - a| &< \frac{\varepsilon}{2}, \end{aligned}$$

then

$$\begin{aligned} |b_n - b| &< \frac{\varepsilon}{2|a_n|} = \tilde{\varepsilon}_1 \\ |a_n - a| &< \frac{\varepsilon}{2|b|} = \tilde{\varepsilon}_2. \end{aligned}$$

For the second one, we can now find some N_2 such that it is true. The first one is a little more tricky because the right side contains a_n. That means the quantity on the right is changin as you change n. But we know that $\lim a_n = a$. We know that a_n is convergent, so we know, by an earlier theorem, that it is bounded. That is, $a_n < M$ for all n. So we just replace a_n with M

$$|b_n - b| < \frac{\varepsilon}{2M} = \tilde{\varepsilon}_1.$$

and our inequality stays true. This now gives us an N_1, and we can choose $N = \max(N_1, N_2)$.

We are now ready to write the proof: Let $\varepsilon > 0$. Since $\lim a_n = a$, we can find N_1 such that for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2|b|}$. If $b = 0$, just set $N_1 = 1$. Since $\lim a_n = a$, then by the theorem which states that every convergent sequence is bounded, we know the sequence $\{a_n\}$ is bounded. Choose M such that for all n, $|a_n| < M$. Also, since $\lim b_n = b$, we can find an N_2 such that for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2M}$. Choose $N = \max(N_1, N_2)$. Suppose $n \geq N$. Then

$$\begin{aligned}
|a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n||b_n - b| + |b||a_n - a| \\
&\leq M|b_n - b| + |b||a_n - a| \\
&\leq M\frac{\varepsilon}{2M} + |b|\frac{\varepsilon}{2|b|} \\
&\leq \varepsilon.
\end{aligned}$$

This concludes the proof.

We are now ready to proof the fourth and final statement of the algebraic limit theorem. Observe that

$$\frac{a_n}{b_n} = a_n \frac{1}{b_n}.$$

We know that $a_n \to a$. If we can show that $\frac{1}{b_n} \to \frac{1}{b}$, we are done. We want

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon.$$

Doing some manipulations, we get

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b - b_n}{b_n b}\right| \\
&= \frac{|b - b_n|}{|b_n||b|}.
\end{aligned}$$

This gives us

$$|b_n - b| < \varepsilon|b||b_n|.$$

By the triangle inequality, we can write

$$|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n|.$$

Then

$$|b| - |b - b_n| \leq |b_n|.$$

Note that if

$$|b - b_n| < \frac{|b|}{2},$$

then

$$\frac{|b|}{2} = |b| - \frac{|b|}{2} \leq |b_n|.$$

This implies that

$$|b_n - b| < \varepsilon|b||b_n|.$$

But the right hand side is larger than $\frac{\varepsilon|b|^2}{2}$. So if

$$|b_n - b| < \frac{\varepsilon|b|^2}{2},$$

and

$$|b_n - b| < \frac{|b|}{2},$$

then we are done.

For our proof, we write: By the third statement in the Algebraic Limit Theorem, it suffices to show that $\lim \frac{1}{b_n} = \frac{1}{b}$. Since $\lim b_n = b$, we can find an N such that for all $n \geq N$,

$$|b_n - b| < \min\left(\frac{\varepsilon|b|^2}{2}, \frac{|b|}{2}\right).$$

Then, if $n \geq N$, $|b_n| > \frac{|b|}{2}$ and $|b_n - b| < \frac{\varepsilon|b|^2}{2}$, so

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} < \frac{\frac{\varepsilon|b|^2}{2}}{|b_n||b|} < \frac{\frac{\varepsilon|b|^2}{2}}{\frac{|b|}{2}|b|} = \varepsilon.$$

1.3.3 Order Limit Theorem

Theorem: Order Limit Theorem

Suppose

$$\lim_{n\to\infty} a_n = a, \quad \text{and} \quad \lim_{n\to\infty} b_n = b,$$

then

1. If $a_n \geq 0$ for all n, then $a > 0$.
2. If $a_n \leq b_n$ for all n, then $a \leq b$.
3. If there is a $c \in \mathbb{R}$ such that $a_n \leq c$ for all n, then $a \leq c$. Similarly, if $c \leq b_n$ for all n, then $c \leq b$.

The third statement is just saying that if a sequence is bounded, say above, by c, then the limit is less than or equal to c.

We can write the first statement as $P \implies Q$, where

- P: $a_n \geq 0$ for all n
- Q: $a \geq 0$

We will prove this by contrapositive, so we need to prove that

$$\text{not}(Q) \implies \text{not}(P),$$

where

- not(Q): $a < 0$

- not(P): there exists an n such that $a_n < 0$

If $a < 0$, we will eventually have an n large enough that all further a_n are negative. Suppose $|a_n - a| < -\frac{a}{2}$, then

$$a_n = a_n - a + a \leq |a_n - a| + a < -\frac{a}{2} + a = \frac{a}{2} < 0.$$

For our proof, we write: Since $\lim a_n = a$ and $-\frac{a}{2} > 0$, there is an N such that for all $n \geq N$, $|a_n - a| < \frac{a}{2}$. Then for all $n \geq N$,

$$a_n = a_n - a + a \leq |a_n - a| + a < -\frac{a}{2} + a = \frac{a}{2} < 0.$$

In particular, $a_N < 0$.

We prove the second statement of the Order Limit Theorem as follows: Apply the first statement of the Order Limit Theorem to the sequence $\{b_n - a_n\}_{n=1}^{\infty}$, then $b_n - a_n \geq 0$ for all n. Therefore, by the first statement of the Order Limit Theorem, $\lim(b_n - a_n) \geq 0$, and by the Algebraic Limit Theorem, $\lim(b_n - a_n) = b - a$, so $b \geq a$.

To prove the third statement of the Order Limit Theorem, we write: If $a_n \leq c$ for all n, then $c - a_n \geq 0$ for all n, so $c - a = \lim(c - a_n) \geq 0$ by the second statement of the Order Limit Theorem. Therefore, $c \geq a$. One can do a similar proof for the second part.

1.3.4 Monotone Convergence Theorem

Recall that a convergent sequence must be bounded, but a bounded sequence is not always convergent.

Definition:

A sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_{n+1} \geq a_n$ for all n. A sequence is **decreasing** if $a_{n+1} \leq a_n$ for all n. A sequence is **monotonic** if it's either increasing or decreasing.

Theorem: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it is convergent.

To prove this, we suppose that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded. Then there is some upper bound M. The limit $\lim a_n = a$ is the least upper bound. Let $A = \{a_n : n \in \mathbb{N}\}$. Then A is nonempty and bounded above, so by the axiom of completeness, it has a least upper bound $a = \sup A$. We want to show that $\lim a_n = a$.

Let $\varepsilon > 0$, then $a - \varepsilon < a$, then $a - \varepsilon$ is not an upper bound for A since a is the least upper bound of A. So we may choose n such that $a_N > a - \varepsilon$.

Let $n \geq N$, then by the monotonicity, $a - \varepsilon < a_N \leq a_n$, but since a is an upper bound for A, we have $a_n < a$. So $a_n \in (a - \varepsilon, a] \subseteq (a - \varepsilon, a + \varepsilon)$. So $|a_n - a| < \varepsilon$ as desired. The proof for the monotonically decreasing case is analogous.

Definition: Convergence of a Series

If $\{a_n\} = a_1, a_2, a_3, \ldots$ is a sequence with the partial sum

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n,$$

then the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots = \lim_{n \to \infty} s_n,$$

provided that this limit exists.

In other words, a series converges if the sequence of its partial sums converges. The sum of the series, is then the limit of the sequence of partial sums.

Rarely will a series converge to a known value, so we prove that $\{s_m\}$ converges in a different way.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

We want to prove that this series converges, so we start by looking at the partial sums

$$s_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2}.$$

Is the sequence $\{s_m\}$ of partial sums bounded and monotone? Since the terms are all positive ($1/n^2$), each $s_{m+1} > s_m$, so we know it is monotone. Is it bounded? We need to show there exists some $M > 0$ such that $s_m < M$ for all $m \in \mathbb{N}$. To do this, we'll rearrange our partial sum in some way so that stuff cancels. We can do this for partial sums, but not necessarily for infinite sums.

$$\begin{aligned} s_m &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2} \\ &\leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right). \end{aligned}$$

In the last line, we broke each term of the second line into a pair of terms using partial fraction decomposition. Now the partial sum telescopes, and we are left with

$$s_m = 2 - \frac{1}{m} < 2.$$

So all of our s_m are bounded by 2. By the monotone convergence theorem, our sequence of partial sum converges, which implies that our series converges. When you're showing boundedness, you typically use a telescoping partial sum as in this example.

Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

To show that a series is divergent, we need to show that the sequence of partial sums is unbounded as $m \to \infty$. In this case, we have

$$\begin{aligned} s_m &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{m} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{m}. \end{aligned}$$

Let $m = 2^k$ for some $k > 0$, then

$$s_{2^k} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2},$$

where there are a total of k terms of $\frac{1}{2}$. Then

$$s_{2^k} = 1 + \frac{1}{2}k.$$

If $k \to \infty$, we see that $s_{2^k} \to \infty$.

We would write our proof as: Let $l \in \mathbb{N}$ such that $l \geq 2^k$. Then $s_l \geq s_{2^k} \geq 1 + \frac{1}{2}k$, so $\{s_l\}$ is unbounded, therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

is divergent.

Theorem: Cauchy Condensation Test

Suppose $\{b_n\}$ is a decreasing sequence and $b_n \geq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} b_n$ converges if and only if

$$\sum_{n=1}^{\infty} 2^k b_{2^k} = b_1 + 2b_2 + 4b_4 + \cdots.$$

converges.

We will prove the half of this theorem that says that convergence of the series above implies the convergence of the series $\sum b_n$.

Assume that $\sum_{n=1}^{\infty} 2^k b_{2^k}$ is convergent. We want to show that $\sum_{n=1}^{\infty} b_n$ is convergent. We need to show that the sequence of partial sums $\{s_m\}$ is bounded and monotone. Since $b_n \geq 0$, we know that $\{s_m\}$ is monotone. We can show that $s_m \leq M$ using the fact that $\sum_{n=1}^{\infty} 2^k b_{2^k}$ is convergent. Since the $b_n \geq 0$, for all $n > 0$, we know that

$$\begin{aligned} s_m &= b_1 + b_2 + b_3 + b_4 + \cdots + b_m \\ &\leq b_1 + b_2 + b_2 + b_4 + b_4 + b_4 + b_4 + \cdots + b_m \\ &= b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + b_m. \end{aligned}$$

Let $m = 2^{k+1} - 1$, then

$$s_{2^{k+1}-1} \leq 1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k,$$

where

$$t_k = 1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}.$$

But we know that $\{t_k\}$ converges, therefore, $\{s_{2^{k+1}-1}\}$ converges. So if l is an integer smaller than $s_{2^{k+1}-1}$, we know that

$$s_l \leq s_{2^{k+1}-1} \leq M,$$

where M is the bound on $\{t_k\}$. Therefore, $\{s_l\}$ converges.

Corollary: p-series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

converges if and only if $p > 1$.

This is a corollary of the Cauchy Condensation Test.

1.3.5 Subsequences

Definition: Subsequence

If $\{a_n\}$ is a sequence of real numbers and $n_1 < n_2 < n_3 < \cdots$ is a list of natural numbers, then

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

is a subsequence of $\{a_n\}$.

Notice that a subsequence contains some subset of the elements of the original sequence, and those elements must be in the same order in the subsequence.

A common way to denote a subsequence is to say that $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Consider the sequence

$$\{a_n\} = \left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Then $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ is a subsequence of $\{a_n\}$, and we could denote it by

$$\{a_{n_k}\}_{k=1}^{\infty} = \{a_{2^{k-1}}\}_{k=1}^{\infty}.$$

Another subsequence of the same sequence is

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{25}, \ldots\right\}.$$

Theorem:

The subsequences of a convergent sequence all converge to the same limit as the original sequence.

Useful consequences of this theorem include

1. Suppose you have a sequence, but you don't know if it converges or diverges. If you find two different subsequences that have different limits, then the sequence must diverge.
   That is, if $\{a_{n_k}\}$ and $\{a_{n_l}\}$ are two different sequences of $\{a_n\}$ and $a_{n_k} \to A$ as $k \to \infty$, and $a_{n_l} \to B$ and $A \neq B$, then $\{a_n\}$ diverges.
2. If $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and $|\{a_{n_k}\}| \to \infty$, that is, it is unbounded as $k \to \infty$, then $\{a_n\}$ diverges.

To prove this theorem, we start by assuming we are given a sequence $\{a_n\}$ that converges to a. We want, for any subsequence $\{a_{n_k}\}$ to converge to the same a. We know that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. We want $|a_{n_k} - a| < \varepsilon$. If $k \geq N$, then $n_k \geq N$. That is, k increases at least as fast as n_k by the definition of a subsequence. Then $|a_{n_k} - a| < \varepsilon$ since $n_k \geq N$. This completes the proof.

Consider the sequence

$$\{a_n\} = (-1)^n = -1, 1, -1, 1, -1, \ldots.$$

We see that the subsequence

$$\{a_{2n-1}\} = -1, -1, -1, -1, -1, \ldots,$$

converges to -1. A second subsequence

$$\{a_{2n}\} = 1, 1, 1, 1, 1, \ldots,$$

converges to 1. Since two subsequences converge to different values, we know by the theorem above that $\{a_n\}$ diverges.

If we know that a sequence converges, the theorem can help you figure out what it converges to. Suppose $0 < b < 1$ and $\{a_n\}$ is the sequence given by $a_n = b^n$. Does $\{a_n\}$ converge? If so, what does it converge to? We can show convergence using the monotone convergence theorem. We can see that $\{a_n\}$ is decreasing, i.e. monotone, and it is bounded between 0 and 1. Suppose $\{a_n\}$ converges to l

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b^n = l.$$

Then the subsequence $\{a_{2n}\} = b^{2n}$ must also converge to l

$$\lim_{n\to\infty} b^{2n} = l.$$

We can write this as

$$\lim_{n\to\infty} (b^n)^2 = \left(\lim_{n\to\infty} b^n\right)^2 = l^2.$$

Now, we have that $l = l^2$, so there are only two possibilities. Either $l = 0$ or $l = 1$. We know it's $l = 0$ because the sequence starts at less than 1 and is monotonically decreasing.

Recall that

$$\text{convergence} \implies \text{bounded},$$

but

$$\text{bounded} \not\Rightarrow \text{convergence}.$$

However, we can strengthen the hypothesis using the monotone convergence theorem

$$\text{bounded} + \text{monotone} \implies \text{convergence}.$$

or, with the Bolzano-Weierstrass theorem, we can give a weaker conclusion

$$\text{bounded} \implies \text{convergent subsequence}.$$

Theorem: Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

The main use of this theorem is that if you have a convergent subsequence $\{a_{n_k}\}$ of a bounded sequence $\{a_n\}$ and $\{a_{n_k}\} \to A$, then $\{a_n\}$ also converges to A provided that $\{a_n\}$ does converge.

Suppose we tell a computer to spit out random numbers in $[-1, 1]$. This means the sequence spit out by the computer is bounded. Then the Bolzano-Weierstrass theorem implies that this sequence of random numbers contains a convergent subsequence. It's not clear how one might find that subsequence.

The idea of the proof is that, since $\{a_n\}$ is bounded by some M, we know that all infinitely many of its terms lie in the interval $[-M, M]$. We now break this interval in half to get two new intervals $[-M, 0]$ and $[0, M]$. Since the original sequence has infinite terms, we know that at least one of these smaller intervals will contain infinitely many terms. We choose a smaller interval with infinitely many terms and repeat the process—breaking it into two subintervals and choosing one with infinitely many terms. We repeat this process indefinitely, and we will have a sequence of intervals that is narrowing down to a certain value. If we select one element from every one of these nested intervals, we have a convergent sequence.

To write the proof more formally, we write:

Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. By boundedness, we can choose an $M < \infty$ such that for all n, we have $a_n \in [-M, M]$.

If $\{n : a_n \in [-M, 0]\}$ is infinite, let $I_1 = [-M, 0]$ and $n_1 = \min\{n : a_n \in [-M, 0]\}$. Otherwise, we know that

$\{n : a_n \in [0, M]\}$ is infinite, then let $I_1 = [0, M]$, and $n_1 = \min\{n : a_n \in [0, M]\}$.

Next, define I_1^L to be the left half of interval I_1, and I_1^R to be the right half of I_1.

Then, if $\{n : a_n \in I_1^L\}$ is infinite, let $I_2 = I_1^L$ and $n_2 = \min\{n : n > n_1, a_{n_2} \in I_2\}$. Otherwise, we know that $\{n : a_n \in I_1^R\}$ is infinite, then let $I_2 = I_1^R$, and $n_2 = \min\{n : n > n_1, a_{n_2} \in [0, M]\}$.

Suppose we have chosen

$$[-M, M] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k,$$

with each $I_j = I_{j-1}^L$ or $I_j = I_{j-1}^R$ and $n_1 < n_2 < n_3 < \cdots < n_k$ and $\{n : a_n \in I_j\}$ is infinite for each j. Then if $\{n : a_n \in I_k^L\}$ is infinite, let $I_{k+1} = I_k^L$. Otherwise $\{n : a_n \in I_k^R\}$ is infinite, so let $I_{k+1} = I_k^R$.

Set $n_{k+1} = \min\{n : n > n_k, a_n \in I_{k+1}\}$. Continuing this construction gives a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ and subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. By the nested interval property, there is an $a \in \bigcap_{j=1}^{\infty} I_j$.

Now we just have to show that

$$a = \lim_{k \to \infty} a_{n_k}.$$

Let $\varepsilon > 0$. Choose M large enough so that $2^{-M} M < \varepsilon$. Then for all $m \geq N$, both a and $a_{n_m} \in I_m \subseteq I_N$. So $|a - a_{n_m}| \leq \text{length}(I_N) = 2 \cdot 2^{-N} \cdot M < \varepsilon$.

1.3.6 The Cauchy Criterion

Think of Cauchy sequences as sequences that *want* to converge.

Definition: Cauchy Sequence

A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists an N such that for all m and n with $m \geq N$ and $n \geq N$, we have $|a_m - a_n| < \varepsilon$.

That is, as you move along the sequence, the terms come closer and closer together.

Recall from the definition of convergence that $\lim a_n = a$ if for every $\varepsilon > 0$, there is an N such that for all $n \geq N$, $|a_n - a| < \varepsilon$.

Theorem:

Every convergent sequence is a Cauchy sequence.

To prove this, we let $\{a_n\}_{n=1}^{\infty}$ be convergent and choose a such that $a = \lim a_n$. We want to prove that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon > 0$. Now, we have to find our N. By the triangle inequality

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So applying $a = \lim a_n$, we can find N such that for all $n \geq N$, $|a_n - a| < \frac{\varepsilon}{2}$. Now if $m \geq N$ and $n \geq N$, then $|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

We can write the definition of a Cauchy sequence symbolically as

$$\forall_{\varepsilon > 0} \exists_N \forall_{n,m \geq N} \; |a_n - a_m| < \varepsilon.$$

Conversely, the symbolic representation of not Cauchy could be written as

$$\exists_{\varepsilon > 0} \forall_N \exists_{n,m \geq N} \; |a_n - a_m| \geq \varepsilon.$$

Lemma:

Every Cauchy sequence is bounded.

To prove this, suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Applying "Cauchyness" with $\varepsilon = 1$, we can find an N such that for all $m, n \geq N$, $|a_m - a_n| < 1$. Now we set $m = N$ and try to bound a_n with $n \geq N$.

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|.$$

Let $M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_n|)$. If $n < N$, then $|a_n| \leq M$ by $M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_n|)$. On the other hand, if $n \geq N$, then $|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N| \leq M$. Thus, for all n, we have $a_n \leq M$.

Theorem:

Every Cauchy sequence is convergent.

To prove this, we let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. By the lemma given above, we know that $\{a_n\}_{n=1}^{\infty}$ is bounded, so by the Bolzano-Weierstrass theorem, it has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. Let $a = \lim a_{n_k}$.

Now we just need to show that our Cauchy sequence also converges to a. That is, we want to show that $\lim a_n = a$.

Let $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we can find an N such that for all $n, m \geq N$, we have $|a_n - a_m| < \frac{\varepsilon}{2}$. Since $\lim a_{n_k} = a$, we can find a K such that for all $k \geq K$, we have $|a_{n_k} - a| < \frac{\varepsilon}{2}$.

Suppose $n \geq N$. We want to show that $|a_n - a| < \varepsilon$. Choose $k = \max(N, K)$, then $|a_{n_k} - a| < \frac{\varepsilon}{2}$ since $k \geq K$. Also, $n_k \geq k \geq N$, so $|a_n - a_{n_k}| < \frac{\varepsilon}{2}$ (using $m = n_k$). Thus, $|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This concludes the proof.

Theorem: Cauchy Criterion

A sequence is convergent if and only if it's a Cauchy sequence.

This is just the combination of the previous two theorems.

1.3.7 Infinite Series

See the earlier section for a definition of an infinite series and its convergence.

Theorem: Algebraic Limit Theorem for Series

Suppose

$$\sum_{j=1}^{\infty} a_j = A, \qquad \text{and} \qquad \sum_{j=1}^{\infty} b_j = B,$$

then

1. For every $c \in \mathbb{R}$,
$$\sum_{j=1}^{\infty} ca_j = cA.$$
2. $$\sum_{j=1}^{\infty}(a_j + b_j) = A + B.$$

While it is not true that $\sum a_j b_j = AB$, there is a "summation by parts" formula for obtaining the sum $\sum a_j b_j$.

The proof of the first part of the Algebraic Limit Theorem for Series is as follows. Write

$$t_n = ca_1 + ca_2 + ca_3 + \cdots + ca_n.$$

We need to show that

$$\lim_{n\to\infty} t_n = cA.$$

Notice that

$$\begin{aligned} t_n &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= cs_n, \end{aligned}$$

Since $\sum a_j = A$, we know that $\lim_{n\to\infty} s_n = A$. Therefore, by the Algebraic Limit Theorem for *Sequences*,

$$\lim_{n\to\infty} cs_n = \lim_{n\to\infty} t_n = cA.$$

The second part can be proved in exactly the same way.

Theorem: Cauchy Criterion for Series

The infinite series $\sum_{j=1}^{\infty} a_j$ converges if and only if for every $\varepsilon > 0$, there exists an N such that for all $n > m \geq N$, we have $|a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$.

To prove this, we start by writing

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

Then $\sum a_j$ converges if and only if $\{s_n\}$ is convergent. By the Cauchy criterion for sequences, this happens if $\{s_n\}$ is a Cauchy sequence. That is, for every $\varepsilon > 0$, there exists an N such that for all $n > m \geq N$, we have $|s_n - s_m| < \varepsilon$.

Note, we could say $n, m \geq N$, but we don't care about the case $n = m$, so we let n be the larger of the two.

Now, if we have

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m,$$

with $n > m$, then

$$s_n - s_m = a_{m+1} + a_{m+1} + \cdots + a_n.$$

Theorem: Divergence Test

If $\sum_{j=1}^{\infty} a_j$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

To prove this, we apply the Cauchy criterion for series with the special case $m = n - 1$. We see for every $\varepsilon > 0$, there exists an N such that for all $n > N$, we have $|a_n - 0| < \varepsilon$.

The usefulness of this theorem is that it can be used as a test for divergence. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.

The converse of this theorem is not true. If $\lim a_n = 0$, it does not imply that $\sum a_n$ is convergent. As an example, consider the harmonic series.

Theorem: The Comparison Test

If you have sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ with $0 \leq a_n \leq b_n$ for all n, then

1. If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
2. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

Notice that the second part of the comparison test is simply the contrapositive of the first, so we only need to prove the first part.

To prove the first part, we suppose that $\sum_{n=1}^{\infty} b_n$ is convergent. Then by the Cauchy criterion for series, for every $\varepsilon > 0$, there exists an N such that for all $n > m \geq N$, we have $|b_{m+1} + b_{m+1} + \cdots + b_n| < \varepsilon$.

Now,

$$\begin{aligned} |a_{m+1} + a_{m+1} + \cdots + a_n| &= a_{m+1} + a_{m+1} + \cdots + a_n \\ &\leq b_{m+1} + b_{m+2} + \cdots + b_n \\ &= |b_{m+1} + b_{m+1} + \cdots + b_n|, \end{aligned}$$

so we have the Cauchy criterion for $\sum_{n=1}^{\infty} a_n$ implying that $\sum_{n=1}^{\infty} a_n$ is also convergent.

Geometric series are special series of the form

$$\sum_{j=0}^{\infty} r^j = 1 + r + r^2 + r^3 + \cdots ,$$

where $r > 0$ has some fixed value. Does this series converge? If so, what does it converge to?

For example, if $r = \frac{1}{2}$, then

$$\sum_{j=0}^{\infty} r^j = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

We can figure out this sum just by thinking about it. Clearly, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. We can use this example geometric series to check that we are using the right formula, which we'll derive soon.

Notice that if $|r| \geq 1$, then $\lim_{j\to\infty} r^j = \infty \neq 0$, so by the divergence test theorem, the series does not converge. On the other hand, if $|r| < 1$, then $\lim_{j\to\infty} r^j = 0$, so the test for divergence does not preclude convergence. These facts hold if r is any complex numbers, i.e., they don't just work for real numbers.

The partial sum of a geometric series is

$$s_n = \sum_{j=0}^{n} r^j = 1 + r + r^2 + r^3 + \cdots + r^n.$$

We can write

$$s_n(1-r) = (1 + r + r^2 + r^3 + \cdots + r^n)(1-r).$$

Expanding the right side, simplifies it to

$$s_n(1-r) = 1 - r^{n+1}.$$

This gives us an explicit formula for the partial sum

$$s_n = \frac{1 - r^{n+1}}{1 - r}.$$

By the algebraic limit theorem for sequences, we get

$$\lim_{n\to\infty} s_n = \frac{1}{1-r}.$$

provided that $|r| < 1$, so

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r},$$

provided that $|r| < 1$.

Another special kind of infinite series is the **p-series**, which is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

We know that the harmonic series ($p = 1$) is divergent. But what about the general case? For $p > 1$, we know the terms go to zero, so convergence is not ruled out by the divergence test.

The Cauchy Condensation Test tells us that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p}$$

converges. We can write this as

$$\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} 2^{n(1-p)} = \sum_{n=1}^{\infty} \left(2^{1-p}\right)^n.$$

This is now a geometric series with $r = 2^{1-p}$, so it will converge if and only if $2^{1-p} < 1$, which occurs when $1 - p < 0$ or $p > 1$.

Definition: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ converges *absolutely* if

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

Theorem: Absolute Convergence Test

If $\sum a_n$ is absolutely convergent, then it is convergent.

To prove this, we suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. It suffices to show that it satisfies the Cauchy criterion. Let $\varepsilon > 0$, then we need to find an N such that if $n > m \geq N$, then $|a_{m+1} + a_{m+1} + \cdots + a_n| < \varepsilon$.

Applying the Cauchy criterion for the convergent series $\sum_{n=1}^{\infty} |a_n|$, we obtain an N such that for all $n > m \geq N$, we have

$$||a_{m+1}| + |a_{m+1}| + \cdots + |a_n|| < \varepsilon.$$

But since all the terms are positive, we can write

$$|a_{m+1} + a_{m+1} + \cdots + a_n| < \varepsilon.$$

By the triangle inequality,

$$|a_{m+1} + a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \varepsilon,$$

so we've shown what we needed to show.

Absolute convergence implies convergence, but convergence does not imply absolute convergence. Consider the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This is convergent, but it's not absolutely convergent because we know that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

is not convergent.

Definition: Conditional convergence

If a series $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is *conditionally* convergent.

Theorem:

If $\{b_n\}_{n=1}^{\infty}$ is a decreasing series of positive numbers, and $\lim_{n\to\infty} b_n = 0$, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n,$$

is convergent.

The convergent alternating harmonic series is an example of this.

Conditionally convergent series are interesting because they can be made to converge to any number if the terms are rearranged appropriately. For example, we know that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent. What if we rearrange it as

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots.$$

This is now a divergent series.

Definition: Rearrangement

If $f : \mathbb{N} \to \mathbb{N}$ is a bijective function and $a_{f(n)} = b_n$ for every n, then $\sum_{n=1}^{\infty} b_n$ is a rearrangement of the series $\sum_{n=1}^{\infty} a_n$.

Theorem:

If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then for every $x \in \mathbb{R}$, there is a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} b_n = x$.

Theorem:

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} b_n$ is any rearrangement, then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

To show whether a series $\sum_{n=1}^{\infty} a_n$ converges or diverges, we can try a few different things.

- If the sequence of partial sums is monotone and bounded, then by the Monotone Convergence Theorem, the series is convergent.
- If the sequence of partial sums is unbounded, then the series diverges.
- Find the limit of the partial sums (i.e. the series converges) or show that it doesn't exist (i.e. the series diverges).
- If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
- Is it a p-series or a geometric series?
- Use the Comparison Test with a p-series or a geometric series.
- If the series converges absolutely, then it converges.
- If $\{a_n\}$ is decreasing, positive, and limits to zero, then the alternating series $\sum (-1)^{n+1} a_n$ converges.

1.4 Topology of the Real Numbers

1.4.1 Open and Closed Sets

Definition: ε-Neighborhood

Given $a \in \mathbb{R}$, and $\varepsilon > 0$, we define the ε-neighborhood of a as

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}.$$

The ε-neighborhood of a is just the interval

$$V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon).$$

Definition: Open set

A set $\mathcal{O} \subseteq \mathbb{R}$ is *open* if for every $a \in \mathcal{O}$, there is an $\varepsilon > 0$, such that $(a - \varepsilon, a + \varepsilon) \subseteq \mathcal{O}$.

In other words, a set is open if you can add or subtract some tiny number to any element of the set, and the result is still in the set. Remember it as, "a set is open if every element of the set has a little wiggle room".

For example, $\mathbb{R}$ is trivially an open set since ε can be anything. The empty set is also trivially an open set.

Any open interval (c, d) is an open set. We can let $\varepsilon = a - c$, then $(a - \varepsilon, a + \varepsilon) = (c, 2a - c)$. So, let $\varepsilon = \min(a - c, d - a)$.

A half-open interval like $(c, d]$ is not an open set. If $a = d$, then for every $\varepsilon > 0$, $(d - \varepsilon, d + \varepsilon)$ is not contained in $(c, d]$.

Theorem:

1. The union of an arbitrary collection of open sets is an open set.
2. The intersection of a finite collection of open sets is an open set.

We can't say the same about the intersection of an arbitrary collection of open sets. For example, consider the infinite collection of open sets $\mathcal{O} = (1 - \frac{1}{n}, 1 + \frac{1}{n})$. Then the intersection $\bigcap_{n\in\mathbb{N}}(1 - \frac{1}{n}, 1 + \frac{1}{n}) = \{1\}$, which is not open.

To prove the first item in this theorem, we let $\{O_\lambda : \lambda \in \Lambda\}$ be any collection of open sets and $\mathcal{O} = \bigcup_{\lambda\in\Lambda} O_\lambda$. We need to show that $\mathcal{O}$ is open. Let $a \in \mathcal{O}$, then there is a λ such that $a \in O_\lambda$. Since O_λ is open, there is an $\varepsilon > 0$, such that $(a - \varepsilon, a + \varepsilon) \subseteq O_\lambda \subseteq \mathcal{O}$. So $\mathcal{O}$ is open.

To prove the second item in this theorem, we suppose that $O_1, O_2, \ldots, O_N$ are open sets and $\bigcap_{j=1} O_j = \mathcal{O}$. We need to show that $\mathcal{O}$ is open. Let $a \in \mathcal{O}$. Then for each j, $a \in O_j$, and since O_j is open, there is an ε_j such that $(a - \varepsilon_j, a + \varepsilon_j) \subseteq O_j$. Then, let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$, then for each j, $\varepsilon \le \varepsilon_j$, so $(a - \varepsilon, a + \varepsilon) \subseteq (a - \varepsilon_j, a + \varepsilon_j) \subseteq O_j$. So $(a - \varepsilon, a + \varepsilon) \subseteq \mathcal{O}$.

Definition: Limit Point

Given $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we say that x is a limit point of A if for every $\varepsilon > 0$, there is an $a \in A$ such that $0 < |a - x| < \varepsilon$.

Essentially, x is a limit point of A if there is an element of A in every ε-neighborhood of x, no matter how large or small that neighborhood is.

Note that the limit point x may or may not be in the set A. Also, notice that $a \neq x$, but the distance between a and x is less than ε.

Theorem:

A point $x \in \mathbb{R}$ is a limit point of $A \subseteq \mathbb{R}$ if and only if there is a sequence $\{a_n\}_{n=1}^{\infty}$ of points in A such that for every n, $a_n \neq x$ and $x = \lim_{n\to\infty} a_n$.

For example, $x = 2$ is a limit point of the set $(1, 2)$, and also of the set $(1, 2) \cup (3, 4)$.

This theorem is an 'if and only if' theorem. To prove the forward implication ($\implies$), we start by supposing that x is a limit point of A. Then for each $n \in \mathbb{N}$, we can apply the definition of the limit point with $\varepsilon = \frac{1}{n}$ to find an $a_n \in A$ with $0 < |a_n - x| < \frac{1}{n}$. Then each $a_n \neq x$ and $\lim_{n\to\infty} a_n = x$.

To prove the reverse implication, we suppose that $x = \lim_{n\to\infty} a_n$ with $a_n \neq x$ and with $a_n \in A$. We need to show that x is a limit point of A. We let $\varepsilon > 0$. Since $x = \lim a_n$, we can find an N such that for all $n \ge N$, $|a_n - x| < \varepsilon$. In particular, $a_N \in A$ and $|a_N - x| < \varepsilon$. Since, by assumption, $a_N \neq x$, we also have $|a_N - x| > 0$, so we are done.

Definition: Isolated Point

A point $a \in A$ is called an isolated point of A if it is not a limit point of A.

Isolated points must be in A, but limit points may or may not be in A. For example, given the set $A = (1, 2)$, then $x = 1.5$ is a limit point since we can construct a sequence like 1.6, 1.55, 1.505, 1.5005,... In fact, every point in this set is a limit point, and no point in this set is an isolated point.

Definition: Closed Set

A set $F \subseteq \mathbb{R}$ is closed if every limit point of F is an element of F.

Consider the set $F = \{1, 2, 7\}$. This set has no limit points. Suppose $x \in \mathbb{R}$ and $x \neq 1$, $x \neq 2$, and $x \neq 7$. Then $\alpha = \min\{|x-1|, |x-2|, |x-7|\} > 0$. Then taking $\varepsilon < \alpha$, there is no $a \in F$ with $|a - x| < \varepsilon$, so x is not a limit point of F.

If F is any finite set, then it has no limit point. In other words, every point in a finite set is an isolated point. This implies that every finite set is a closed set.

Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. None of these points are limit points. Given any $\frac{1}{n} \in A$, the closest point in A to $\frac{1}{n}$ is $\frac{1}{n+1}$, and

$$\left|\frac{1}{n+1} - \frac{1}{n}\right| = \frac{1}{n(n+1)}.$$

Then take $\varepsilon < \frac{1}{n(n+1)}$, and we see that $\frac{1}{n}$ is not a limit point. The only limit point for this set is $x = 0$. This is a limit point since $\lim \frac{1}{n} = 0$ and each $\frac{1}{n} \neq 0$, and each $\frac{1}{n} \in A$.

Theorem: Density of $\mathbb{Q}$ in $\mathbb{R}$

Every $y \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

To prove this, we start with $y \in \mathbb{R}$, and let $\varepsilon > 0$. Recall our earlier theorem about the density of $\mathbb{Q}$ in $\mathbb{R}$. We showed that if $y < z$ are real numbers, then there is an $r \in \mathbb{Q}$ such that $y < r < z$. We apply this theorem with $y = y$ and $z = y + \varepsilon$ to find $r \in \mathbb{Q}$ with $r \in (y, y + \varepsilon)$.

Definition: Closure

Given $A \subseteq \mathbb{R}$, write L for the set of limit points of A. Then the closure of A is $\overline{A} = A \cup L$.

The idea is that $\overline{A}$ is now a closed set. Consider the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. The only limit point for this set is zero, so $L = \{0\}$. So the closure of A is $\overline{A} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

Theorem:

Suppose $A \subseteq \mathbb{R}$, then

1. $\overline{A}$ is a closed set
2. For every closed set $F \supseteq A$, we have $\overline{A} \subseteq F$.

The second item in this theorem is just another way of saying that $\overline{A}$ is the smallest closed set containing A.

To prove the first part of this theorem, we start by supposing that y is a limit point for $\overline{A}$. There *is* something to prove here because we don't know (yet) if $\overline{A}$ has the same limit points as A. Our claim is that y is also a limit point of A. Let $\varepsilon > 0$. Since y is a limit point of $\overline{A}$, there is an $x \in \overline{A}$ with $|x - y| < \frac{\varepsilon}{2}$. Since $x \in \overline{A}$, either $x \in A$ or x is a limit point of A. So if $x \in A$, then $0 < |x - y| < \varepsilon$ and we are done. Otherwise, if $x \notin A$, then x is a limit point of A, so there is an $a \in A$ with $|x - a| < \min\left(\frac{\varepsilon}{2}, |x - y|\right)$. By the triangle inequality,

$$|y - a| = |y - x + x - a| \leq |y - x| + |x - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and $a \neq y$ since $|x - a| < |x - y|$. Since $y \in L$, we have $y \in \overline{A} = A \cup L$, so we are done.

To prove the second part of the theorem, we suppose that F is closed with $A \subseteq F$. We need $A \cup L \subseteq F$. That is, $L \subseteq F$. Suppose y is a limit point of A, then since $A \subseteq F$, we also know that y is a limit point of F. Since F is closed, $y \in F$.

Recall that the complement of a set A is the set of all real numbers that is not in A.

Theorem:

1. $\mathcal{O}$ is open if and only if $\mathcal{O}^c$ is closed
2. F is closed if and only if F^c is open

To prove the forward implication ($\Longrightarrow$) for the first item in this theorem, we suppose $\mathcal{O}$ is an open set. Let y be a limit point of $\mathcal{O}^c$. We want to show that $y \in \mathcal{O}^c$. The contrapositive of this is that $y \in \mathcal{O} \implies y$ is not a limit point for $\mathcal{O}^c$. Let $y \in \mathcal{O}$. Since $\mathcal{O}$ is open, there exists an $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subset \mathcal{O}$. So $(y - \varepsilon, y + \varepsilon) \cap \mathcal{O}^c = \phi$. So there are no points $x \in \mathcal{O}^c$ with $0 < |x - y| < \varepsilon$, so y is not a limit point of $\mathcal{O}^c$.

To prove the reverse implication for the first item, we suppose that $\mathcal{O}^c$ is closed and we let $y \in \mathcal{O}$. Since $y \notin \mathcal{O}^c$ and $\mathcal{O}^c$ is closed, y is not a limit point of $\mathcal{O}^c$. So there is an $\varepsilon > 0$ such that there are no points $x \in \mathcal{O}^c$ with $0 < |y - x| < \varepsilon$. That is, if $0 < |y - x| < \varepsilon$, then $x \in \mathcal{O}$, so $(y - \varepsilon, y) \cup (y, y + \varepsilon) \subseteq \mathcal{O}$. Since $y \in \mathcal{O}$, then $(y - \varepsilon, y + \varepsilon) \subseteq \mathcal{O}$.

To prove the forward implication ($\Longrightarrow$) for the second item in this theorem, we suppose that F is closed. Then $F = (F^c)^c$. By item one of this theorem, this implies that F^c is open.

To prove the reverse implication for the second item in this theorem, we suppose that F^c is open then by item one of this theorem, we have that $(F^c)^c$ is closed. But $F = (F^c)^c$, so F is closed.

Theorem:

1. The intersection of an arbitrary collection of closed sets is closed.
2. The union of a *finite* collection of closed sets is closed.

To prove this theorem, recall De Morgan's laws

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c.\end{aligned}$$

For the first item in this theorem, suppose we have an arbitrary collection of closed sets F_j, then

$$F = \bigcap_j F_j,$$

is the intersection of those sets. De Morgan's laws give us

$$F^c = \left(\bigcap_j F_j\right)^c = \bigcup_j F_j^c.$$

This tells us that the complement of the intersection of an arbitrary collection of closed sets F_j is equal to the union of the sets F_j^c. Since the F_j are closed, the previous theorem tells us that the F_j^c are open. So F^c is a union of open sets and is therefore open itself. So by the previous theorem, F is closed.

For the second item, suppose we have a finite collection of closed sets F_j and

$$F = \bigcup_{j=1}^{N} F_j,$$

is the union of those closed sets. Then De Morgan's laws tell us that

$$F^c = \left(\bigcup_{j=1}^{N} F_j\right)^c = \bigcap_{j=1}^{N} F_j^c.$$

By the previous theorem, since the F_j are closed, we know that the F_j^c are open. So F^c is the intersection of finite open sets, and is itself open. Therefore, since F^c is open, the previous theorem tells us that F is closed.

1.4.2 Compact Sets

Compactness removes some of the problems associated with infinity when dealing with infinite sets.

Definition: Compact set

A set $K \subseteq \mathbb{R}$ is compact if every sequence of points in K has a subsequence that converges to a point in K.

With most authors, this is the definition of "sequential compactness." They define compactness in terms of covers.

Before we start proving that certain sets are compact, the following lemma will help:

Lemma:

If F is closed, and $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence in F, then

$$\lim_{n\to\infty} a_n \in F.$$

That is, the limit of the sequence is also in F.

To prove this lemma, we start with the limit

$$a = \lim_{n\to\infty} a_n,$$

of the sequence. Then there are two cases two consider:

1. If there is an n such that $a_n = a$, then since all $a_n \in F$, we also have $a \in F$.
2. If $a_n \neq a$ for every n, then a must be a limit point of F (by a previous theorem). But since F is closed, $a \in F$ (since a closed set contains all of its limit points).

This concludes our proof of the lemma.

For example, a closed interval $[c, d]$ is compact. To prove this, suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in $[c, d]$. The Bolzano-Weierstrass theorem says that every bounded sequence has a convergent subsequence. We know the set $\{x_n\}_{n=1}^{\infty}$ is bounded (set $M = \max(|c|, |d|)$ then $|x_n| \leq M$ for every n). Therefore, by the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Suppose this subsequence has the limit $x = \lim_{k\to\infty} x_{n_k}$. Then by our lemma, $x \in [c, d]$ since $[c, d]$ is closed.

Definition: Bounded set

A set $A \subseteq \mathbb{R}$ is bounded if there is a finite M such that for every $a \in A$, $|a| \leq M$.

Note, it is possible to have unbounded closed sets. For example, $\mathbb{R}$ and $\mathbb{Z}$ are trivially closed, but they are also unbounded. The set $[0, 1] \cup [2, 3] \cup [4, 5] \cup \ldots$ is also a closed but unbounded set.

Theorem:

A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proving the forward implication ($\implies$) of this theorem, requires two parts

1. K is compact $\implies$ K is closed
2. K is compact $\implies$ K is bounded

To prove the first of these, we assume K is compact. Then we want to prove that K is closed. Let x be a limit point of K, then we need to prove that $x \in K$. Since x is a limit point of K, there is a sequence $\{x_n\}_{n=1}^{\infty}$ in K with $x = \lim_{n\to\infty} x_n$. By compactness, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} \in K$. But $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x$, so now we know that $x \in K$.

We prove the second of these by the contrapositive K unbounded $\implies$ K is not compact. Let K be an unbounded set, then for each $n \in \mathbb{N}$, there is an $x_n \in K$ with $|x_n| > n$. Then $\{x_n\}_{n=1}^{\infty} \subset K$, and every subsequence of $\{x_n\}$ is unbounded ($|x_{n_k}| > n_k > k$), so no subsequences converge, and so K is not compact.

To prove the reverse implication of this theorem, we suppose that K is closed and bounded and $\{x_n\}_{n=1}^{\infty}$ is a sequence in K. Since K is bounded, $\{x_n\}_{n=1}^{\infty}$ is bounded. So by the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Then by our lemma from earlier, $\lim_{k\to\infty} x_{n_k} \in K$.

Theorem:

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ are all compact and nonempty, then their intersection is nonempty.

The intersection, incidentally, is also compact since each K_i is closed and bounded by the fact that it is compact. The intersection is then closed and bounded, and therefore, compact.

This theorem is a generalization of the Nested Interval Property.

To prove this theorem, for each n, let $x_n \in K_n$. Then $\{x_n\}_{n=1}^{\infty} \subset K_1$ by nestedness. Since K is compact, there is a subsequence $\{x_{n_k}\}_{n=1}^{\infty}$ with

$$x = \lim_{k\to\infty} x_{n_k} \in K_1.$$

The claim is that

$$x \in \bigcap_{n=1}^{\infty} K_n.$$

We fix n, then for all $K \geq n$, we also have $n_k \geq n$, so $x_{n_k} \in K_{n_k} \subseteq K_n$. In other words, for $K \geq n$, we have $x_{n_k} \in K_n$. After throwing out the first $n-1$ terms, the remaining terms are all in K_n, so since K_n is closed, the lemma guarantees that $x \in K_n$.

In metric spaces, compact implies closed and bounded, but the converse is not necessarily true. The converse fails in infinite dimensional vector spaces.

Definition: Open Cover

Given $A \subseteq \mathbb{R}$, an open cover of A is any collection of open sets $\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda.$$

Definition: Finite Subcover

Given a collection of sets that forms a cover of A, a finite subcover is a finite subcollection of those sets whose union still covers A.

The question is often, given a cover, can we extract a finite subcover?

Consider the set $A = (0,1)$. An open cover of A is defined by the union

$$A \subseteq \bigcup_{x\in(0,1)} \left(\frac{x}{2}, 1\right).$$

To prove that the union of sets on the right really covers A, suppose y is any element of $A = (0,1)$. Then $0 < \frac{y}{2} < y < 1$. So if $y = x$, then $y \in \left(\frac{x}{2}, 1\right)$. In other words, for all possible $y \in A$,

$$y \in \left(\frac{x}{2}, 1\right) \subseteq \bigcup_{x\in(0,1)} \left(\frac{x}{2}, 1\right),$$

so this union of sets is an open cover of A. However, we cannot find a finite subcover of A. We can take finitely many of these sets, $x_1, x_2, x_2, \dots, x_N$ and order them so that $x_1 < x_2 < \cdots < x_N$. As x gets smaller, the interval $\left(\frac{x}{2}, 1\right)$ gets larger. The intervals are nested. That is,

$$\left(\frac{x_1}{2}, 1\right) \supseteq \left(\frac{x_2}{2}, 1\right) \supseteq \cdots \supseteq \left(\frac{x_N}{2}, 1\right),$$

so the union of these finite sets is

$$\bigcup_{j=1}^{N} \left(\frac{x_j}{2}, 1\right) = \left(\frac{x_1}{2}, 1\right).$$

This means that any element of A that is smaller than $\frac{x_1}{2}$, for example $\frac{x_1}{4}$ is not in this union of sets. Therefore, this finite union of sets does not cover A, so it cannot be a finite subcover.

Now consider the closed set $A = [0,1]$. Note that

$$\bigcup_{x\in(0,1)} \left(\frac{x}{2}, 1\right),$$

does not cover A because it does not include 0 or 1. We can fix this by including two small open sets at each end

$$A = [0,1] \subset \bigcup_{x\in(0,1)} \left(\frac{x}{2}, 1\right) \cup (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon).$$

Now we can extract a finite subcover. Take $x = \varepsilon$, then

$$A = [0,1] \subseteq \left(\frac{\varepsilon}{2}, 1\right) \cup (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon).$$

Tip:

Remember that covers/subcovers are *collections* of sets.

Theorem: Heine-Borel Theorem

Let $K \subseteq \mathbb{R}$. Then the following statements are equivalent.

1. K is compact
2. K is closed and bounded
3. Every open cover of K has a finite subcover

By a previous theorem, we've already proved that the first and second statements in this theorem are equivalent. To finish proving this theorem, we just need to show that either of the first two statements is equivalent to the third. We'll start by showing that the third statement implies the second. Suppose K satisfies the third statement. Consider the open covering

$$K \subseteq \bigcup_{x \in K} (x-1, x+1).$$

That is, every x in K is in the middle of $(x-1, x+1)$, so this is a covering. Since we are assuming the third statement is true, we know that for the open covering given above, there must be a finite subcover. So

$$K \subseteq (x_1 - 1, x_1 + 1) \cup \cdots \cup (x_N - 1, x_N + 1).$$

Now take $M = 1 + \max(|x_1|, \ldots, |x_N|)$, then if $x \in (x_j - 1, x_j + 1)$, then

$$|x| = |x - x_j + x_j| \leq |x - x_j| + |x_j| < 1 + |x_j| \leq M.$$

So the subcover given above must be bounded, which means K is bounded. This is the first part of our proof. The rest is left as an exercise for the poor reader.

Note: # 3 in this theorem is how most authors *define* compactness. This author's definition of compactness is what most other authors call "sequential compactness".

1.5 Functional Limits and Continuity

1.5.1 Functional Limits

So far we've only talked about sequential limits, i.e., the limits like

$$\lim_{n\to\infty} x_n = L,$$

of sequences. We will now look at functional limits, i.e., the limits like

$$\lim_{x\to c} f(x) = L,$$

of functions.

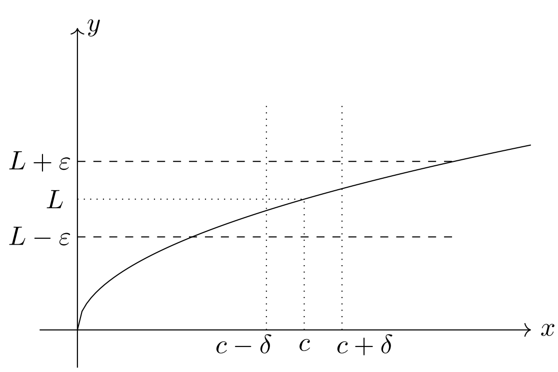
Definition: Functional Limit

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and c is a limit point of A. Then we say

$$\lim_{x\to c} f(x) = L,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$, whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

In other words, a point $f(c) = L$ is the limit of $f(x)$ as $x \to c$ if for any small ε, you can find a small δ such that if your x is in $(x - \delta, x + \delta)$ then your $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$.



If c were not a limit point of A, we could find $\delta > 0$ such that there are no points $x \in A$ with $0 < |x - c| < \delta$ so $\lim_{x\to c} f(x) = L$ is vacuously true for every L.

Example:

Show that

$$\lim_{x\to 3} 4x + 1 = 13.$$

First, we have to do some scratchwork. We want

$$\begin{aligned} |f(x) - 13| &< \varepsilon \\ |4x + 1 - 13| &< \varepsilon \\ |4x - 12| &< \varepsilon \\ |x - 3| &< \frac{\varepsilon}{4}. \end{aligned}$$

So our proof would be written as: Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{4}$, then if $0 < |x - 3| < \delta$, then by our scratchwork, we have $|(4x + 1) - 13| < \varepsilon$.

Proving the existence of the limits of other kinds of functions using this definition can be a bit more challenging. Suppose we want to prove

$$\lim_{x\to 2} x^2 = 4.$$

We want $|f(x) - L| < \varepsilon$. We can start with

$$\begin{aligned} |x^2 - 4| &< \varepsilon \\ |(x - 2)(x + 2)| &< \varepsilon \\ |x - 2||x + 2| &< \varepsilon \\ |x - 2| &< \frac{\varepsilon}{|x + 2|}. \end{aligned}$$

Now, the right side, our δ depends on x. Let's choose a preliminary value for δ, say $\delta = 1$, then if $|x - 2| < 1$, the triangle inequality gives us

$$|x + 2| = |x - 2 + 4| \le |x - 2| + 4 < 5.$$

So

$$\frac{\varepsilon}{|x + 2|} > \frac{\varepsilon}{5}.$$

In other words, if $|x - 2| < \frac{\varepsilon}{5}$ we will also have $|x - 2| < \frac{\varepsilon}{|x+2|}$.

We would write our proof as: Let $\varepsilon > 0$. Take $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$. If $|x - 2| < \delta$, we have $|x - 2| < 1$, so by our scratchwork above, we have $|x + 2| < 5$. Then $|x^2 - 4| = |x - 2||x + 2| \le \delta|x + 2| < 5\delta \le 5\frac{\varepsilon}{5} = \varepsilon$.

Note that our first choice of δ was arbitrary, and then the triangle inequality gave us a second one. We could just as well have used a different preliminary δ.

Theorem:

If $f : A \to \mathbb{R}$ and c is a limit point of A, then the following are equivalent:

1. $$\lim_{x\to c} f(x) = L.$$
2. For every sequence, $\{x_n\}_{n=1}^{\infty} \subseteq A$ which converges to c, and for all n, $x_n \neq c$, we have
$$\lim_{n\to\infty} f(x_n) = L.$$

This theorem is sometimes called the **Sequential Criterion for Functional Limits**. It tells us that if we have a sequence whose x-values are converging to c, then the $f(x)$ values are converging to L.

To prove that the first statement implies the second, we suppose that $\lim_{x\to c} f(x) = L$, and $\{x_n\}_{n=1}^{\infty} \subseteq A$ with $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. We want to show that $\lim_{n\to\infty} f(x_n) = L$.

Let $\varepsilon > 0$. We need N such that for all $n \geq N$, $|f(x_n) - L| < \varepsilon$. From $\lim_{x\to c} f(x) = L$, we can find a $\delta > 0$, such that if $0 < |x-c| < \delta$, we have $|f(x)-L| < \varepsilon$. By $\lim_{n\to\infty} x_n = c$, there's an N such that for all $n \geq N$, we have $0 < |x_n - c| < \delta$. Now we just have to combine the two. Putting these together, if $n \geq N$, then $0 < |x_n - c| < \delta$, so $|f(x_n) - L| < \varepsilon$ as desired.

To prove that the second statement in this theorem implies the first, we do the contrapositive. That is, we show that the negation of the first statement implies the negation of the second statement.

Suppose the first statement is not true, then there exists an $\varepsilon > 0$, such that for every $\delta > 0$, there is an $x \in A$, with $0 < |x - c| < \delta$ but $|f(x) - L| \geq \varepsilon$. We to to use this to prove the negation of the second statement. Now, for each n, we apply the above with $\delta = \frac{1}{n}$ to find x_n with $0 < |x_n - c| < \frac{1}{n}$ but $|f(x) - L| \geq \varepsilon$. Then $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ but $\lim_{n\to\infty} f(x) = L$.

Corollary: Algebraic Functional Limit Theorem

Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, c is a limit point of A, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$. Then

1. $$\lim_{x\to c} kf(x) = kL, \quad \text{for every } k \in \mathbb{R}$$
2. $$\lim_{x\to c} [f(x) + g(x)] = L + M$$
3. $$\lim_{x\to c} f(x)g(x) = LM$$
4. $$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$

To prove this corollary, we combine the Algebraic Limit Theorem for sequences with the earlier theorem. For example, to prove the third case, that the limit of the product of two functions is the product of the limits of the two functions, then by the Sequential Criterion for Functional Limits, it suffices to show that for every $\{x_n\}_{n=1}^{\infty} \subseteq A$ with $x_n \neq c$, and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n)g(x_n) = LM$.

Let $\{x_n\}$ be such a sequence, then by the S.C.F.L., since $\lim_{x\to c} f(x) = L$, we have $\lim_{n\to\infty} f(x_n) = L$. Similarly, $\lim_{n\to\infty} g(x_n) = M$. Then by the A.L.T. for sequences, we have $\lim_{n\to\infty} f(x_n)g(x_n) = LM$. This is exactly what we wanted to prove.

The rest of the statements in this corollary are proved the same way—just using different statements from the A.L.T. for sequences.

Corollary: Divergence Criterion

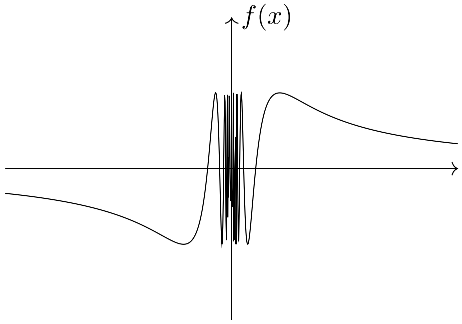
Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. If $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $\{y_n\}_{n=1}^{\infty} \subseteq A$ with $x_n \neq c$ and $y_n \neq c$ and $c = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to c} f(x)$ does not converge or it does not exist.

This corollary is the **Divergence Criterion for Functional Limits**.

To prove this, we want to show that $\lim f(x_n) \neq \lim g(x_n)$ implies that $\lim f(x)$ does not converge. This is equivalent to the contrapositive statement that $\lim_{x\to c} f(x)$ converges implies that $\lim f(x_n) = \lim g(x_n)$. This is immediate from the S.C.F.L.

We can use this corollary to prove that
$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$
does not exist. To do that, we look for two sequences that

give different limits.



Since $\sin\left(\frac{1}{x}\right)$ is hitting $f(x)=0$ faster and faster as $x\to 0$, we can construct a sequence. Setting $x_n=\frac{1}{n\pi}$, we have $\lim_{n\to\infty} x_n = 0$, and we have

$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = 0.$$

But $\sin\left(\frac{1}{x}\right)$ is also hitting $f(x)=1$ faster and faster as $x\to 0$. Setting

$$y_n = \frac{1}{\frac{\pi}{2}+2n\pi},$$

we have $\lim_{n\to\infty} y_n = 0$, and we have

$$\lim_{n\to\infty} \sin\left(\frac{1}{y_n}\right) = 1.$$

Therefore, by our corollary, the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not converge.

1.5.2 Continuous Functions

Definition: Continuous Functions

Suppose $f: A\to\mathbb{R}$ and $c\in A$. Then f is continuous at c if for every $\varepsilon>0$, there exists a $\delta>0$ such that for all $x\in A$ with $|x-c|<\delta$, we have $|f(x)-f(c)|<\varepsilon$.

Note that while c is in A, it may or may not be a limit point of A.

"Continuous on A" means f is continuous at c for every c in A.

There are two facts of note

1. If c is not a limit point of A then f is automatically continuous at c, by the definition. That is, it's trivially continuous at c.
2. If c is a limit point of A then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Theorem:

f is continuous at c if and only if for every sequence $\{x_n\}_{n=1}^{\infty}\subseteq A$ with $\lim_{n\to\infty} x_n = c$, we have

$$\lim_{n\to\infty} f(x_n) = f(c).$$

This theorem is sometimes called the **Sequential Characterization of Continuity**.

Theorem: Algebraic Continuity Theorem

Suppose $f: A\to\mathbb{R}$, $g: A\to\mathbb{R}$, $c\in A$, and f and g are continuous at c. Then

1. kf is continuous at c for every scalar k
2. $f+g$ is continuous at c
3. fg is continuous at c
4. $\frac{f}{g}$ is continuous at c provided $g(c)\neq 0$.

The proof of this is immediate from the Algebraic Functional Limit Theorem. If c is an isolated point, it is automatically true. Otherwise, for example,

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c).$$

The first equality comes from the Algebraic Functional Limit Theorem, and the second equality holds because f and g are continuous.

We want a large family of functions that we know are continuous. We can get to a lot of them by starting with $f(x)=x$.

To prove that $f(x)=x$ is continuous on $\mathbb{R}$, we let $c\in\mathbb{R}$, then we want f to be continuous at c. We want $|f(x)-f(c)|<\varepsilon$. This implies that $|x-c|<\varepsilon$. But $|x-c|<\delta$, so we can take $\delta=\varepsilon$.

We write the proof as: Let $\varepsilon>0$. Taking $\delta=\varepsilon$, if $|x-c|<\delta$ then $|f(x)-f(c)|=|x-c|<\delta=\varepsilon$, so $|f(x)-f(c)|<\varepsilon$ as desired.

Using the Algebraic Continuity Theorem, we can show that all polynomials and rational functions are continuous. For example, to show that $f(x)=x^n$ is continuous for all integers n, we would use induction. Recall that induction works as follows:

1. Show that the statement is true for $n=0$
2. Show that if it's true for n, it must also be true for $n+1$.

For $f(x)=x^0=1$, if given $\varepsilon>0$, we can take an arbitrary δ like $\delta=3$, then $|f(x)-f(c)|=|1-1|<\varepsilon$. For the second step in induction, we suppose x^n is continuous at c, then we need to show that this implies that $f(x)=x^{n+1}$ must be continuous at c. We can write $f(x)=x^{n+1}=x\cdot x^n$. But we know x is continuous at c by the previous example. We know that x^n is continuous at c by the induction hypothesis. So by the Algebraic Continuity Theorem, the product $x \cdot x^n$ is continuous at c.

To show that polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$ is continuous, we would apply the second statement in the Algebraic Continuity Theorem $n-1$ times, using the fact that $a_j x^j$ is continuous which we deduce from the first statement in the theorem as well as the previous example which showed that x^j is continuous.

Similarly, we can show that rational functions

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n},$$

are continuous provided that $f(c)$ is defined.

We can even extend this to real analytic functions, which are really just infinitely long polynomials. For example,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

Theorem: Composition of Continuous Functions

Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with Range$(f) \subseteq B$, we can define the function composition $f \circ g : A \to \mathbb{R}$ by

$$g \circ f(x) = g\left(f(x)\right).$$

Then if $c \in A$ and f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

To prove this, we use the definition of continuity. Let $\varepsilon > 0$, then we need a $\delta > 0$ such that for all $x \in A$, if $|x - c| < \delta$, we have $|g(f(x)) - g(f(c))| < \varepsilon$.

From the continuity of g at $f(c)$, we can find an $\eta > 0$ such that for all $y \in B$ with $|y - f(c)| < \eta$, we have $|g(y) - g(f(c))| < \varepsilon$.

From the continuity of f at c with η in place of ε, we can obtain a $\delta > 0$ such that for all $x \in A$ with $|x-c| < \delta$, we have $|f(x) - f(c)| < \eta$.

Now if $|x-c| < \delta$, we apply the continuity of g at $f(c)$ with $y = f(x)$, then we see that $|g(f(x)) - g(f(c))| < \varepsilon$ as desired.

Consider again the function

$$f(x) = \sin\left(\frac{1}{x}\right),$$

which has the domain $A = \mathbb{R}\backslash\{0\}$. We know from earlier work that $\lim_{x\to 0} f(x)$ does not converge. This implies that $f(x)$ is not continuous at 0 since 0 is a limit point of the domain. For $c \neq 0$, $\frac{1}{x}$ is continuous at $x = c$ since it is just a rational function. That is, $\frac{1}{x}$ is continuous except at $x = 0$ and $\sin y$ is continuous on $\mathbb{R}$, so the composed function is also continuous everywhere but at $x = 0$.

Consider the function

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

If $c \neq 0$, $f(x)$ is continuous since $\sin\left(\frac{1}{x}\right)$ is continous and x is continuous, so the product of the two is also continuous. If $c = 0$, $f(x)$ is still continuous. To show this, we have to go back to the definition of continuity. Let $\varepsilon > 0$. We need $\delta > 0$ such that if $|x| < \delta$ then $|f(x)| < \varepsilon$. Taking $\delta = \varepsilon$, then $|f(x)| \leq |x| < \delta = \varepsilon$.

1.5.3 Continuous Functions on Compact Sets

Given a function $f : A \to \mathbb{R}$ and $B \subseteq A$, then we define

$$f(B) = \{f(x) : x \in B\},$$

to be the "image of B under f". Similarly, we define

$$f^{-1}(B) = \{x \in A : f(x) \in B\},$$

to be the "preimage of B". Note that f is continuous if and only if for all open sets B, $f^{-1}(B)$ is open.

A general question is, if B satisfies some property and f is continuous, does $f(B)$ also satisfy that property? In other words, which properties of sets are preserved under continuous mappings?

Instance 1: If B is open is $f(B)$ necessarily open? Not necessarily. A counterexample is $A = \mathbb{R}$, $f(x) = 1$ for all x. Let $B = \mathbb{R}$, then B is open, but $f(B) = \{1\}$ is closed. So continuous functions do not preserve openness.

Instance 2: What about closedness? The answer is that closedness is not necessarily preserved by continuous functions. Consider the function

$$f(x) = \frac{1}{1+x^2}, \qquad A = \mathbb{R}.$$

Let $B = \mathbb{R}$, then $f(B) = (0, 1]$, which is not a closed set. For example, zero is a limit point, but is not in the set.

Instance 3: What about compactness? Recall that a set is compact if every sequence of points in the set has a convergent subsequence which converges to a point in the set. The answer is that, yes, compactness is preserved under continuous mappings.

Theorem: The Preservation of Compact Sets

If $f : A \to \mathbb{R}$ is continuous, and $K \subseteq A$ is compact, then $f(K)$ is compact.

To prove this, we start by supposing f is continuous and K is compact. Given a sequence of points in $f(K)$, we need to show it contains a convergent subsequence that converges to a point in $f(K)$.

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $f(K)$. We need to find a convergent subsequence which converges to a point in $f(K)$. For each n, since $y_n \in f(K)$, there is a point $x_n \in K$ such that $f(x_n) = y_n$. Now we have a sequence in K. Then $\{x_n\}_{n=1}^{\infty} \subseteq K$. Since K is compact, this has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} x_{n_k} = x \in K.$$

Then

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x).$$

The last equality comes from the Sequential Characterization of Continuity and the fact that f is continuous at x. Now, since $x \in K$, $f(x) \in f(K)$, and we are done.

Corollary: Extreme Value Theorem

If $f : K \to \mathbb{R}$ is continuous and K is compact, then f attains a maximum and minimum value on K. That is, there exist $x_0, x_1 \in K$ such that for all $x \in K$, $f(x_0) \leq f(x) \leq f(x_1)$.

Next, we prove this. We know by Preservation of Compact Sets, $f(K)$ is compact, so $f(K)$ is closed and bounded. Since it is bounded, we know $\alpha = \sup(f(K))$ exists. There are two possibilities

1. α is not a limit point of $f(K)$
2. α is a limit point of $f(K)$

If α is a limit point of $f(K)$, then $\alpha \in f(K)$ since $f(K)$ is closed. If α is not a limit point of $f(K)$, then we can choose ε so that $(\alpha-\varepsilon, \alpha+\varepsilon)$ contains no points in $f(K)$ other than α. If $\alpha \notin f(K)$, then $\alpha - \varepsilon/2$ would be an upper bound for $f(K)$. This is a contradiction, so we must have $\alpha \in f(K)$. For either case, then, we have $\alpha \in f(K)$.

So we can choose x_1 with $f(x_1) = \alpha$. Then for all $\alpha \in K$, $f(x) \leq f(x_1)$ since α is an upper bound. To find the x_0, we can use an analogous argument with inf instead of sup.

Uniform Continuity

Before we talk about the difference between continuity and *uniform* continuity, let us look at a few examples.

Consider the function $f(x) = 3x+1$ where $f : \mathbb{R} \to \mathbb{R}$. We know that f is continuous on $\mathbb{R}$. Let $c \in \mathbb{R}$, and let $\varepsilon > 0$. Then $|3x + 1 - (3c + 1)| < \varepsilon$. This gives us $3|x - c| < \varepsilon$, which implies

$$|x - c| < \frac{\varepsilon}{3},$$

so we can take $\delta = \frac{\varepsilon}{3}$ to show continuity. So $f(x)$ is continuous at c, but δ does not depend on c.

Next, consider the function $f(x) = x^2$. We again have that f is continuous on $\mathbb{R}$. We again let $c \in \mathbb{R}$, and let $\varepsilon > 0$. Then $|x^2 - c^2| = |x - c||x + c| < \varepsilon$, which implies

$$|x - c| < \frac{\varepsilon}{|x + c|}.$$

Note that $|x + c| = |x - c + 2c| \leq |x - c| + 2|c|$. If we choose $\delta = 1$, then $|x + c| \leq 1 + 2|c|$, so

$$\frac{\varepsilon}{|x + c|} \geq \frac{\varepsilon}{1 + 2|c|}.$$

Taking

$$\delta = \frac{\varepsilon}{1 + 2|c|},$$

then if $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. But now, δ depends on c.

The idea of uniform continuity is that we can use one δ for every point instead of needing a different δ for different values of c.

Definition: Uniform Continuity

A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $x, y \in A$ with $|x-y| < \delta$, we have $|f(x)-f(y)| < \varepsilon$.

Symbolically, we would write this definition as

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in A}\forall_{\substack{y\in A \\ |x-y|<\delta}} \ |f(x) - f(y)| < \varepsilon.$$

Recall that the definition of regular continuity is written symbolically as

$$\forall_{x\in A}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{\substack{y\in A \\ |x-y|<\delta}} \ |f(x) - f(y)| < \varepsilon.$$

The only difference between the two definitions is the location of $\forall_{x\in A}$.

Note: A function is (or not) continuous at a *point*. A function is (or not) uniformly continuous on a *set*.

Theorem:

A function $f : A \to \mathbb{R}$ is *not* uniformly continuous on A if and only if there exists a $\varepsilon_0 > 0$ and sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in A such that

$$\lim_{n\to\infty}(x_n - y_n) = 0,$$

but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

This theorem is called the **Sequential Criterion for the Absence of Uniform Continuity**.

To prove this, we note that not(uniformly continuous) is equivalent to

$$\text{not}\left(\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{\substack{x,y\in A\\|x-y|<\delta}}\ |f(x)-f(y)|<\varepsilon\right).$$

This is equivalent to

$$\exists_{\varepsilon_0>0}\forall_{\delta>0}\exists_{\substack{x,y\in A\\|x-y|<\delta}}\ |f(x)-f(y)|\geq\varepsilon_0.$$

So if f is not uniformly continuous, there is an $\varepsilon_0>0$ such that

$$\forall_{\delta>0}\exists_{\substack{x,y\in A\\|x-y|<\delta}}\ |f(x)-f(y)|\geq\varepsilon_0.$$

Applying this fact for each n with $\delta=\frac{1}{n}$, we can find $x_n, y_n\in A$ with $|x_n-y_n|<\frac{1}{n}$ but $|f(x_n)-f(y_n)|\geq\varepsilon_0$. Then $\lim_{n\to\infty}(x_n-y_n)=0$, but $|f(x_n)-f(y_n)|\geq\varepsilon_0$. Now we are done with proving that direction of the iff. The other direction is pretty straightforward.

Theorem: Uniform Continuity on Compact Sets

If K is a compact set, and f is continuous on K, then f is uniformly continuous on K.

The proof is left as an exercise.

Note, $\mathbb{R}$ is closed, but unbounded, so it is not compact. This theorem, then, does not work on $\mathbb{R}$.

1.5.4 Intermediate Value Theorem

Connected Sets

Recall that two sets A and B are disjoint if $A\cap B=\phi$. Now we will define something stronger than disjointness.

Definition: Separated Sets

Two nonempty sets A and B are separated if $A\cap\overline{B}=\phi$ and $\overline{A}\cap B=\phi$.

Recall that $\overline{B}$ is the closure of B which is the set B plus its limit points. Then $A\cap\overline{B}=\phi$ means A is disjoint from the closure of B. This is stronger than $A\cap B=\phi$ because $\overline{B}$ is generally a larger set than B.

Consider the sets $A=(0,1)$ and $B=[1,2)$. These sets are disjoint but not separated. The closure of A is $\overline{A}=[0,1]$, so $\overline{A}\cap B=\{1\}\neq\phi$.

As another example, consider the sets $A=(0,1)$ and $B=(1,2)$. These sets are disjoint and separated.

Definition: Disconnected Sets

E is disconnected if there are nonempty sets A and B which are separated and $E=A\cup B$.

In other words, the union of separated sets is a disconnected set.

Definition: Connected Sets

E is connected if it is not disconnected.

There are two important facts that we must note.

Fact: A set of real numbers a and b is connected if and only if it is an interval (a,b), $[a,b]$, $(a,b]$, or $[a,b)$.

Fact: E is an interval if and only if whenever $x,y\in E$ and $x<c<y$, we also have $c\in E$. That is, a set is an interval if and only if for any pair of numbers in the set, every number between those two is also in the set.

Theorem: Connectedness by Sequences

A set $E\subseteq\mathbb{R}$ is connected if and only if for each pair of nonempty disjoint sets A and B with $A\cup B=E$, there exists a convergent sequence $\{x_n\}_{n=1}^{\infty}$ with either

1. $$\{x_n\}_{n=1}^{\infty}\subseteq A \quad\text{and}\quad \lim_{n\to\infty}x_n\in B$$
2. or $$\{x_n\}_{n=1}^{\infty}\subseteq B \quad\text{and}\quad \lim_{n\to\infty}x_n\in A$$

By contrapositive, this is the same as showing that "not connected" if and only if not this theorem.

Not connected means the same as disconnected. A set is disconnected if and only if there exists nonempty, disjoint sets A and B with $E=A\cup B$ and $A\cap\overline{B}=\phi$ and $\overline{A}\cap B=\phi$.

Not this theorem is the same as saying there exist nonempty, disjoint sets A and B with $E=A\cup B$ such that for all convergent sequences $\{x_n\}_{n=1}^{\infty}\subseteq A$, we have $\lim_{n\to\infty}x_n\not\subseteq B$ and for all convergent sequences $\{x_n\}_{n=1}^{\infty}\subseteq B$, we have $\lim_{n\to\infty}x_n\not\subseteq A$.

We see that these mean the same thing. There was no proof required–just an unraveling of the definitions.

Theorem: Connectedness by Intervals

A set $E\subseteq\mathbb{R}$ is connected if and only if for all $a,b\in E$ and c satisfying $a<c<b$, we also have $c\in E$.

To prove the forward direction of the implication, we suppose E is connected and we have $a, b \in E$ and $a < c < b$. We want to show that this implies $c \in E$.

We write $A = E \cap (-\infty, c)$, which are all the points in E strictly less than c. We write $B = E \cap (c, \infty)$, which are all the points in E strictly greater than c.

We know these are nonempty since $a \in A$ and $b \in B$. Furthermore, A and B are separated because $\overline{A} \subset \overline{(-\infty, c)} = (-\infty, c]$ but $(-\infty, c] \cap B = \phi$. Similarly, $\overline{B} \subset [c, \infty) \cap A = \phi$.

If $A \cup B = E$, then E is disconnected, which would be a contradiction. So $A \cup B \neq E$, from which we can conclude that $c \in E$. This ends our proof for the forward direction of the implication.

To prove the reverse direction of the implication, we suppose that E satisfies "for all $a, b \in E$ and c satisfying $a < c < b$, we also have $c \in E$". That is, E is an interval. Suppose $E = A \cup B$ with $A \cap B = \phi$ and A and B are nonempty. We need to show that $\overline{A} \cap B \neq \phi$ or $A \cap \overline{B} \neq \phi$.

Since A and B are nonempty, we can choose $a_0 \in A$ and $b_0 \in B$. Without loss of generality, we can assume that $a_0 < b_0$. Since a_0 and b_0 are in E, by the property "for all $a, b \in E$ and c satisfying $a < c < b$, we also have $c \in E$", we know that $[a_0, b_0] \subset E$. We will call this interval I_0.

For the midpoint of I_0, we have $\frac{a_0+b_0}{2} \in E$, so $\frac{a_0+b_0}{2}$ is in A or B. Let $I_1 = [a_1, b_1]$ be either the left half or right half of I_0 so that $a_1 \in A$ and $b_1 \in B$. Repeating this forever, we get

$$[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$$

By the nested interval property, we know that

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is nonempty since the length of $[a_n, b_n]$ approaches zero as $n \to \infty$. So

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x\},$$

for some $x \in E$. If $x \in A$, then $\lim_{n\to\infty} b_n = x \in A$, but $\{b_n\} \subseteq B$. If $x \in B$, then $\lim_{n\to\infty} a_n = x \in B$, but $\{a_n\} \subseteq A$. Then by a previous theorem, we are done.

Theorem: Preservation of Connected Sets

Suppose $f : G \to \mathbb{R}$ is continuous on G and $E \subseteq G$ is connected. Then $f(E)$ is connected.

To prove this theorem, we want to show that $f(E)$ is connected using the Connectedness by Sequences theorem. Let A and B be nonempty and disjoint with $f(E) = A \cup B$. Let $C = \{x \in E : f(x) \in A\}$. That is, $f^{-1}(A)$. Let $D = \{x \in E : f(x) \in B\}$. Then C and D are disjoint and nonempty, and $E = C \cup D$. Since E is connected, we can apply the Connectedness by Sequences theorem to obtain a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ with $\lim_{n\to\infty} x_n \in D$ (or vice versa).

Then $\{f(x_n)\}_{n=1}^{\infty} \subseteq A$ and since f is continuous,

$$\lim_{n=1} f(x_n) = f\left(\lim_{n\to\infty} x_n\right),$$

and $\lim_{n\to\infty} x_n \in D$, so $\lim_{n\to\infty} f(x_n) \in B$ (or vice versa).

Intermediate Value Theorem

Theorem: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there is a $c \in (a, b)$ with $f(c) = L$.

Basically, the intermediate value theorem tells us that you can graph a continuous function without lifting your pencil.

The intermediate value theorem is a very important one, and we will prove it in three different ways. The first uses connectedness.

Proof 1: The idea is that it suffices to show that $f([a, b])$ is an interval.

By the Connectedness by Intervals theorem, this means showing that $f([a, b])$ is connected. We know $[a, b]$ is an interval, and by this theorem, we know $[a, b]$ is connected. Then by the Preservation of Connected Sets theorem, we know that $f([a, b])$ is connected.

So the intermediate value theorem is just a special case of the theorem of Preservation of Connected Sets.

Proof 2: For the second proof, we use the Axiom of Completeness, which tells us that every nonempty set of real numbers that is bounded above has a least upper bound.

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$. Let $E = \{x \in [a, b] : f(x) < L\}$. We know E is nonempty since $f(a) < L$ and $a \in [a, b]$. E is bounded above. So by the axiom of completeness, $c = \sup(E)$ exists.

We need to show that $f(c) = L$. We have three possibilities:

1. $f(c) < L$
2. $f(c) = L$
3. $f(c) > L$

Suppose $f(c) < L$. Let $\varepsilon = f(c) - L > 0$. By continuity, we can find a $\delta > 0$ such that if $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon = f(c) - L$. In particular, $f(c+\varepsilon/2) < L$. But then $c + \frac{\varepsilon}{2} \in E$ is a contradiction since c is the least upper bound for E.

Suppose $f(c) > L$. Let $\varepsilon = f(c) - L$. Find δ so that if $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. In particular, $f(x) > L$ for all $x \in [c - \varepsilon/2, c]$. So $c - \frac{\varepsilon}{2}$ is an upper bound for E, contradicting the condition that c is the least upper bound for E.

The only possibility we have left is that $f(c) = L$.

Proof 3: The third proof of the intermediate value theorem uses the nested interval property. That is, if $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ are closed intervals, then

$$\bigcap_{n=0}^{\infty} I_n,$$

is nonempty.

Assume $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$.

Write $I_0 = [a,b] = [a_0, b_0]$. So $f(a_0) < L < f(b_0)$. Next, we look at the midpoint of I_0. If

$$f\left(\frac{a_0+b_0}{2}\right) = L,$$

then we are done. If

$$f\left(\frac{a_0+b_0}{2}\right) < L,$$

set

$$a_1 = \frac{a_0+b_0}{2}, \qquad b_1 = b_0.$$

Then we have $f(a_1) < L$ and $f(b_1) > L$. Then set $I_1 = [a_1, b_1]$. Else, if

$$f\left(\frac{a_0+b_0}{2}\right) > L,$$

set

$$a_1 = a_0, \qquad b_1 = \frac{a_0+b_0}{2}.$$

Then we have $f(a_1) < L < f(b_1)$. Then set $I_1 = [a_1, b_1]$.

Continue this process indefinitely. Either we eventually find some c with $f(c) = L$, or we find a sequence

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

with $I_n = [a_n, b_n]$ and $f(a_n) < L < f(b_n)$ and $b_n - a_n = 2^{-n}(b-a)$ (i.e. they are shrinking). But the nested interval property implies that

$$\bigcap_{n=0}^{\infty} I_n,$$

is nonempty, and by the fact that the lengths of the intersections are shrinking, the intersection must be the single point $\{c\}$.

We have $\lim_{n\to\infty} a_n = c$. By continuity,

$$\lim_{n\to\infty} f(a_n) = f(c).$$

Since $f(a_n) < L$, then $f(c) \le L$. Similarly, we have $\lim_{n\to\infty} b_n = c$. By continuity,

$$\lim_{n\to\infty} f(b_n) = f(c).$$

Since $f(b_n) > L$, then $f(c) \ge L$.

Theorem: Zeros of Odd Polynomials

Consider an odd degree polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

with n odd and $a_n \ne 0$ and $a_0, \ldots, a_n \in \mathbb{R}$. Then there is a point $x_0 \in R$ with $f(x_0) = 0$.

In other words, all odd polynomials have at least one zero. Proving this theorem shows a typical application of the Intermediate Value Theorem.

The idea for the proof is that the dominant term in the polynomial is $a_n x^n$. For $|x|$ large enough, $a_n x^n$ should overwhelm the other terms, meaning $f(x)$ has the same sign as $a_n x^n$. In particular, we should be able to find N large enough such that $f(N) > 0$ and $f(-N) < 0$ if $a_N > 0$ or vice versa if $a_N < 0$. f is continuous on $[-N, N]$, so the intermediate value theorem guarantees a $c \in (-N, N)$ with $f(x) = 0$. Now, we just need to make this idea more concrete.

We need

$$N > 1 + \frac{2n \max\left(|a_0|, |a_1|, \ldots, |a_{n-1}|\right)}{|a_n|}.$$

It turns out this is big enough to make $a_N x^N$ overwhelm the other terms.

Suppose $j < n$, then

$$|a_j| \le \max\left(|a_0|, |a_1|, \ldots, |a_{n-1}|\right) < \frac{N|a_n|}{2n},$$

so

$$|a_j N^j| \le \frac{1}{2n}|a_n| N N^j \le \frac{1}{2n}|a_n N^n|.$$

Then

$$|f(N) - a_n N^n| = \left|\sum_{j=0}^{n-1} a_j N^j\right| \le \sum_{j=0}^{n-1} |a_j N^j|,$$

by the triangle inequality. So

$$|f(N) - a_n N^n| \le \sum_{j=0}^{n-1} \frac{1}{2n}|a_n N^n| = \frac{1}{2}|a_n N^n|.$$

So $f(N)$ has the same sign as $a_n N^n$. We can do the same thing for $f(-N)$.

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