My name is Leon Hostetler. I am currently a student at Florida State University majoring in physics as well as applied and computational mathematics. Feel free to download, print, and use these class notes. If you find them useful, consider buying me a coffee.

All of my class notes can be found at www.leonhostetler.com/classnotes

Please bear in mind that these notes will contain errors. If you find one, please email me at leonhostetler@gmail.com with the name of the class notes, the page on which the error is found, and the nature of the error. If you include your name, I will probably list your name on the thank you page if I decide to compile and sell my notes.

This work is currently licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. That means you are free to copy and distribute this document in whole for noncommericial use, but you are not allowed to distribute derivatives of this document or to copy and distribute it for commerical reasons.

Last Updated: August 25, 2016

Chapter 1

Trigonometry

1.1 Angles, Degrees, and Special Triangles

What are the characteristics of the 30-60-90 triangle? The hypotenuse is two times the length of the short side, and the other side is the square root of three times the short side. Note, a 30-60-90 triangle is half of an equilateral triangle.

What are the characteristics of the 45-45-90 triangle? The shorter sides are equal in length, and the hypotenuse is the square root of two times the short side. This is easy to figure out once you realize that this is an isosceles triangle.

> **Example:**
>
> Find the remaining sides of a 30-60-90 triangle if the longest side is 6.
>
> The ratio of sides in a 30-60-90 triangle is $1 : \sqrt{3} : 2$. If the longest side is 6, then we find the other two sides by multiplying them by 3 so that the sides are $3 : 3\sqrt{3} : 6$.
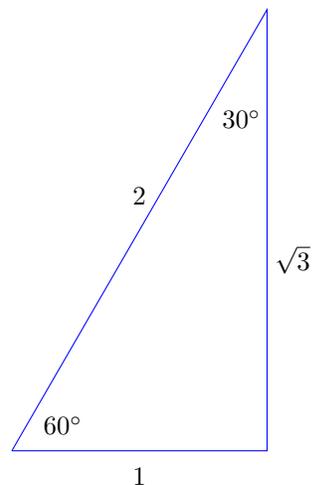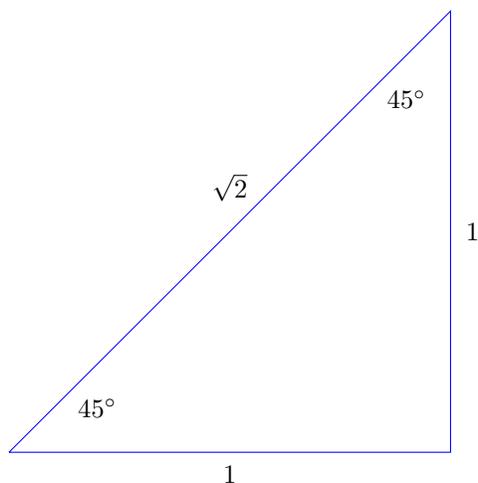
1.2 The Rectangular Coordinate System

On the Cartesian coordinate system, the distance between two points (x_1, y_1) and (x_2, y_2) is given by the **distance formula**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is simply an application of the Pythagorean theorem.

What is the unit circle? The unit circle is a circle of radius 1 that is centered on the origin. Its equation is $x^2 + y^2 = 1$. To check if a point is on the unit circle, use the distance function with the given point and the origin $(0, 0)$. If the point is on the unit circle then the distance will be one.

What is an angle in standard position? Its vertex is at the origin, and its initial side is the positive x-axis.

How do you find a point on the **terminal side** of an angle that is in standard position? Convert the terminal side of the angle to an equation from which you can determine the slope of the line.

An angle is **coterminal** with a given angle if it has the same terminal side. For example, the angles $30°$ and $30° + 360°$ are coterminal.
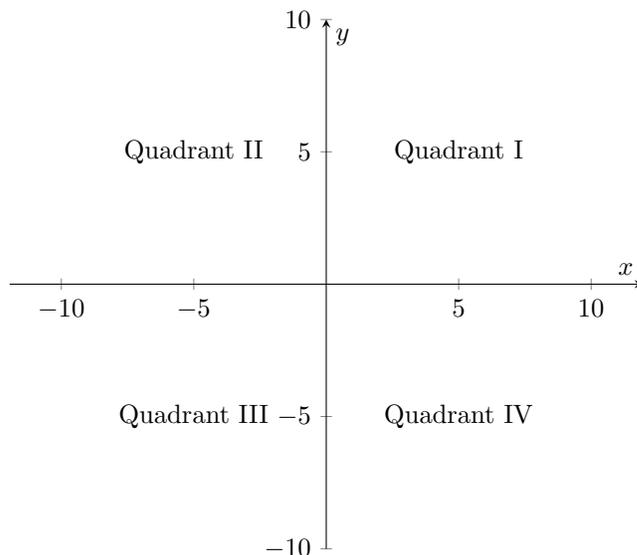
1.3   Trigonometric Functions

The trigonometric functions relate the sides of a right triangle to its angles.

To remember the trigonometric functions, memorize the mnemonic **SOH-CAH-TOA**, from the first letters of

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \text{SOH}$$
$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \text{CAH}$$
$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \text{TOA}.$$

The other three functions can be remembered as the reciprocals of the above functions

$$\cot \theta = \frac{1}{\tan \theta}$$
$$\sec \theta = \frac{1}{\cos \theta}$$
$$\csc \theta = \frac{1}{\sin \theta}.$$

Notice that since tangent is opposite over adjacent, it is a measure of the slope of the hypotenuse. In other words, the tangent of an angle is the slope of the terminal side of the angle.

Example:

Given that $\csc \theta = \frac{5}{3}$ and $\cos \theta < 0$, find the remaining trigonometric functions.

Recall that $\csc \theta$ is the inverse of $\sin \theta$, so $\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}$. We are given that $\csc \theta = \frac{5}{3}$, so we can refer to a generic right triangle with hypotenuse of 5 and an opposite side of length 3. Since it is positive and the hypotenuse is always positive, then in this case the opposite side (length 3) must also be positive, which implies that the angle is either in quadrant I or quadrant II. Since $\cos \theta$ is negative and cosine is adjacent over hypotenuse, this implies that the adjacent angle must be negative, which means our angle must be in quadrant two. Using the Pythagorean theorem, we find that the adjacent side must be -4. Now that we know all the sides, we can easily find all the trigonometric functions.

$$\sin \theta = \frac{3}{5}$$
$$\cos \theta = -\frac{4}{5}$$
$$\tan \theta = -\frac{3}{4}$$
$$\cot \theta = -\frac{4}{3}$$
$$\sec \theta = -\frac{5}{4}$$

1.4   Identities

The **reciprocal identities** are

$$\sin \theta = \frac{1}{\csc \theta}$$
$$\cos \theta = \frac{1}{\sec \theta}$$
$$\tan \theta = \frac{1}{\cot \theta}$$
$$\cot \theta = \frac{1}{\tan \theta}$$
$$\sec \theta = \frac{1}{\cos \theta}$$
$$\csc \theta = \frac{1}{\sin \theta}.$$

The **ratio identities** are

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$
$$\cot \theta = \frac{\cos \theta}{\sin \theta}.$$

The **Pythagorean identities** are

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$1 + \tan^2 \theta = \sec^2 \theta$$
$$1 + \cot^2 \theta = \csc^2 \theta.$$

The second two can be derived by dividing the first one by $\cos \theta$ and by $\sin \theta$.

Note that $\sin^2 \theta$ means $(\sin \theta)^2$.

> **Example:**
>
> Simplify $(\cos \theta - \sin \theta)^2$.
>
> FOILING it out, we get $\cos^2 \theta - 2 \cos \theta \sin \theta + \sin^2 \theta$. Using the identity $\sin^2 \theta + \cos^2 \theta = 1$, we can rewrite this as $1 - 2 \cos \theta \sin \theta$.

1.5   Right Triangle Trigonometry

Sine and cosine are **cofunctions**, and so are tangent and cotangent and secant and cosecant. The trigonometric function of an angle is always equal to the cofunction of the **complement** of the angle. The complement of an angle θ is $90° - \theta$. What this means is that if two angles $A + B = 90°$, then

$$\sin A = \cos B$$
$$\sec A = \csc B$$
$$\tan A = \cot B.$$

In other words, if you have a right triangle, then you know that the two small angles A and B sum to $90°$. From the above, you know that the sine of one of the small angles will equal the cosine of the other small angle, and so on.

The **supplement** of an angle θ is $180° - \theta$.

Using the special 30-60-90 and 45-45-90 right triangles, we can easily deduce the following trigonometric functions for the angles $30°$, $45°$, and $60°$.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Know how to enter degrees, minutes, and seconds into the calculator (Use 2nd > Angle), and how to convert between decimal/minute/second (DMS) and decimal degrees (DD).

Know how to do the inverse trig functions on a calculator. For example, $\sin^{-1}(45°)$.

Note: The $\sin^{-1}$ and related keys do not give the reciprocals of the sine and related, rather, they give the result of the inverse function of sine and related. In other words, if you have the sine value and need to know what degree it corresponds to, use this key.

> **Example:**
>
> Given the right triangle ABC with C $= 90°$, $a = 68$ and $b = 104$, find the unknown side and angles.
>
> Remember that by convention, c is the side opposite the angle C, and so on.
>
> The first step when solving a triangle problem is to diagram it and note the known information. In this case, we have the two legs of the triangle, so we use the Pythagorean theorem to find the unknown side, $c = \sqrt{104^2 + 68^2} = 124.3$. Next we can use the tangent function to find the angle A since
>
> $$\tan A = \frac{68}{104},$$
>
> which implies that
>
> $$A = \tan^{-1} \left(\frac{68}{104} \right) = 33.2°.$$
>
> Finally, we find angle B by using the fact that the interior angles of a triangle sums to $180°$.

> **Example:**
>
> In what quadrant does the angle θ terminate if $\cos\theta < 0$ and $\csc\theta > 0$.
>
> The first step is to draw a quick coordinate system just to help the reasoning process. We are told that the cosine of the angle is negative and since cosine is adjacent over hypotenuse then the adjacent side must be negative (hypotenuse is always positive). This eliminates quadrants I and IV where the adjacent side would be positive. Finally, we are told that the cosecant of the angle is positive. Since cosecant is the reciprocal of sine and sine is opposite over hypotenuse, then cosecant is hypotenuse over opposite. We are told that the cosecant of the angle is positive, which implies that the opposite side is positive (hypotenuse is always positive). This eliminates quadrant III. The only quadrant remaining is quadrant II.

1.6   Vectors

An angle measured from the horizontal up is called the **angle of elevation**. An angle measured from the horizontal down is called the **angle of depression**.

The **bearing** of a direction is the small angle between the line and the north-south direction. To denote a bearing, we first give N or S to indicate whether the bearing is more generally north or south, then we give the angle, and finally the direction east or west to indicate which side of the north-south direction it is. For example, N 30° E means the direction that is 30° to the east of straight north.

A **scalar** is a regular number with only magnitude. A **vector** is a mathematical object that contains both a magnitude and a direction. For example, 30 miles straight north is a vector because it has a direction (straight north) and a magnitude (30 miles). On a coordinate system, a vector is represented as an arrow pointing in a specific direction. The length of the arrow is the magnitude of the vector and the direction of the vector can be specified by the angle between the arrow and the horizontal. The **magnitude** of a vector is simply the length of the vector.

To denote that a quantity is a vector, we use a large bold letter such as **V** or a large bold letter with an arrow on top $\vec{V}$. The magnitude of a vector is denoted by putting it between vertical bars $|\vec{V}|$.

The sum of vectors is called the **resultant vector**. To sum vectors, set the one vector's tail to the other vector's head. The resultant vector is the one from the tail of the first to the head of the second. To subtract vectors, set both tails together and the resultant vector is the one from head to head. Alternatively, you can add the two but reverse the direction of the one being subtracted.

A vector in **standard position** has its tail at the origin of the Cartesian coordinate system. Given a vector, in standard position, with a specified magnitude and degrees it is easy to use trig functions to break the vector into its horizontal and vertical vector components.

For a vector $\vec{V}$, its magnitude and its x and y components are given by

$$\begin{aligned} V_x &= |\vec{V}|\cos\theta \\ V_y &= |\vec{V}|\sin\theta \\ |\vec{V}| &= \sqrt{V_x^2 + V_y^2} \end{aligned}$$

where θ is the angle between the vector in standard position and the positive x-axis.

You can use the calculator to find vector components, and the resultant vector given its components. Use the features under **2nd** > **Angle**.

- **P ▶ Rx**, input (magnitude, angle) to receive the x-component (i.e. horizontal component of the vector
- **P ▶ Ry**, input (magnitude, angle) to receive the y-component (i.e. vertical component of the vector
- **R ▶ Pr**, input (x-component, y-component) to receive the magnitude of the resultant vector
- **R ▶ P** θ, input (x-component, y-component) to receive the angle of the resultant

In physics, a **force** is often represented by a vector since forces have both magnitude and direction.

When an object is not moving, it is said to be in **static equilibrium**. When an object is in static equilibrium, it means that all the forces on it cancel, so that there is no net force. If there was a net force on the

object, then it would be moving. To solve static equilibrium problems treat each force on the object as a separate vector. Then choose a convenient coordinate system and break each vector into x-components and y-components. Then since the object isn't moving, all the x components sum to zero and all the y-components sum to zero, so you can solve for this missing information by using a system of equations.

When it comes to static equilibrium problems remember that **weight** is a force—it is mass times the gravitational acceleration.

Work is a physics term defined as the force $\vec{F}$ applied to an object times the distance d that the force causes the object to move

$$W = |\vec{F}|d\cos\theta,$$

where θ is the angle between the force and the object's resulting direction of motion.

Note: The physics definition of work has very little in common with the common definition of work. For example, there is work involved in moving an extremely light object a very tiny distance, but there is no work involved in holding an extremely heavy object at some height for a time.

A vector in standard position with the endpoint of (a, b) can be represented as

$$\vec{V} = \langle a, b \rangle.$$

This is called the **component form** because a is the horizontal component and b is the vertical component. The magnitude of $\vec{V}$ can easily be found using the Pythagorean theorem on a and b.

Vectors of this form can be added and subtracted simply by adding/subtracting their components.

Vectors of this form can be multiplied by a scalar simply by multiplying each of the components by the scalar.

A unit vector is any vector with a magnitude of 1. The unit vector along the x-axis is called the unit horizontal vector and is denoted by $\hat{i}$ and the unit vector along the y-axis is called the unit vertical vector and is denoted by $\hat{j}$.

Any vector can be written as the sum of the above unit vectors. For example

$$\vec{V} = \langle 1, 2 \rangle = \hat{i} + 2\hat{j}.$$

Writing vectors in this form is called **vector component form**. Vectors of this form can be added, subtracted, and multiplied by scalar multiples in the same way as vectors in component form.

The **dot product** or **scalar product** is a special product of two vectors. Given the vectors $\vec{A} = \langle a_x, a_y \rangle$ and $\vec{B} = \langle b_x, b_y \rangle$ that have an angle θ between them, their dot product is

$$\begin{aligned} \vec{A} \bullet \vec{B} &= a_x b_x + a_y b_y \\ \vec{A} \bullet \vec{B} &= AB\cos\theta. \end{aligned}$$

Notice that the dot product of two vectors is a scalar, that is, it's a regular number. The dot product is often used to find the angle between two vectors.

1.7 Reference Angle

A reference angle is a related angle. It is the positive acute angle between the terminal side of an angle θ in standard position and the x-axis. The trig functions for any angle will be the same as the one for its reference angle except maybe for a change in sign.

When using the inverse trig functions to find the angle that gives the trig values, always use positive values. This is because reference angles are always positive acute angles. Then with other given information, determine in

which quadrant the reference angle is and generate the appropriate real angle. When finding an angle using the inverse trig functions, there may be more than one possible angle that fits the data. Check all four quadrants to see if you would get the same result there. It always helps to draw a quick coordinate system and the given angle. To learn why the inverse values entered must be positive, I need to wait until the inverse trig functions are covered in the text. Always test your answer.

1.8 Radians and Degrees

A radian is the arc length that is equal to the circle's radius. Since a radian is a ratio of two lengths, it is a unitless measure.

To convert an angle into radians, divide the arc length by the radius. One radian is approximately $57.3°$.

Since $360° = 2\pi$ radians you can use the conversion factor

$$\frac{180°}{\pi \text{ rad}} \qquad \text{or} \qquad \frac{\pi \text{ rad}}{180°}.$$

To convert degrees to radians on the calculator, the calculator must be in radian mode. Then simply enter

the degree and the degree symbol (under **2nd** > **AN-GLE**) and press enter. To convert radians to degrees, the calculator must be in degree mode. Then simply enter the angle in radians and the r symbol (under **2nd** > **ANGLE**) and press enter.

1.9   Circular Functions
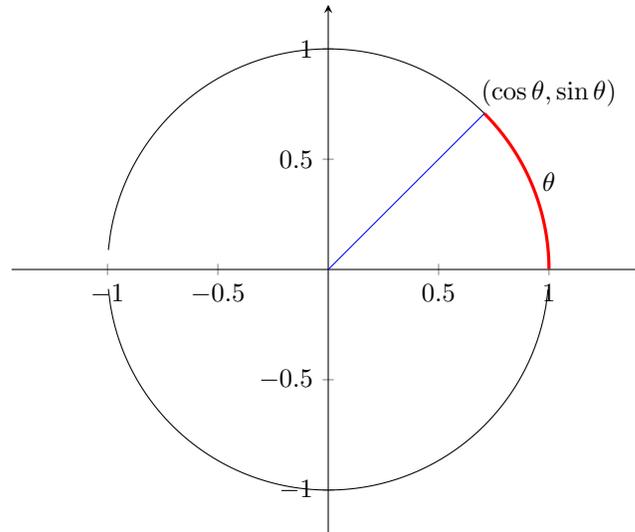
The **unit circle** is a circle of radius 1 that is centered at the origin. The unit circle possesses some properties that make it useful when dealing with trigonometric coordinates. For example, the x value of any point on the unit circle is $\cos\theta$ where θ is the angle that goes through that point. Similarly, the y value of any point on the unit circle is $\sin\theta$. Also, if the angle θ is in radians, then the arc length (of the circle) that is between the initial and terminal sides of the angle θ is exactly θ. In other words, radians is not only a measure of angle—it is also the distance along the unit circle contained by a given angle.

The unit circle can be graphed on a graphing calculator by switching from degree mode to radian mode and from function mode to parametric mode then inputting $\cos(T)$ for X and $\sin(T)$ for Y.

The **great circle distance** is the distance between two points on the surface of a large sphere such as Earth.

1.10   Arc Length and Area of a Sector

If θ (in radians) is the central angle in a circle with radius r, then the length of the arc s cut off by the angle is

$$s = r\theta.$$

The area of the sector of a circle is given by

$$A = \frac{1}{2}r^2\theta,$$

provided that θ is measured in radians.

Example:

For a clock, how far does the tip of the minute hand travel in 43 minutes if it is 4 inches long?

Using the equation for arc length $s = r\theta$ and the conversion factor $\frac{2\pi}{60\,\text{min}}$, we get

$$s = (4\,\text{in}) \cdot 43\,\text{min} \cdot \frac{2\pi}{60\,\text{min}} = 18\,\text{in}.$$

Remember that when using this formula, the angle must be in radians! When doing this kind of problem, check your work against the circumference. In this case, the circumference of the clock is $2\pi 4\,\text{in} = 25\,\text{in}$. Since our minute hand swept out approximately $\frac{2}{3}$ of the circumference, our answer should be approximately $\frac{2}{3} \cdot 25\,\text{in} = 16.7\,\text{in}$, and it is.

1.11 Characteristics of the Trig Functions
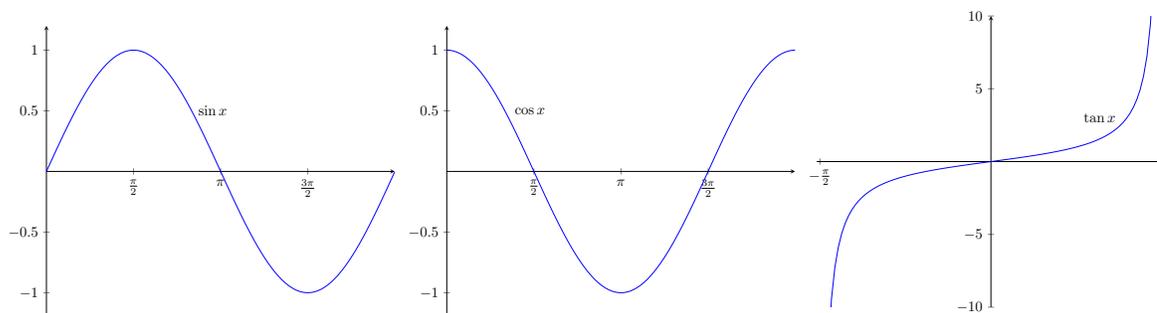
	Domain	Range	Period	Amplitude
$\sin x$	$\mathbb{R}$	$[-1, 1]$	2π	1
$\cos x$	$\mathbb{R}$	$[-1, 1]$	2π	1
$\tan x$	$\left\{x \,\middle|\, x \neq \frac{\pi}{2} + k\pi\right\}$	$\mathbb{R}$	π	N/A
$\cot x$	$\left\{x \,\middle|\, x \neq k\pi\right\}$	$\mathbb{R}$	π	N/A
$\sec x$	$\left\{x \,\middle|\, x \neq \frac{\pi}{2} + k\pi\right\}$	$(-\infty, 1] \cup [1, \infty)$	2π	N/A
$\csc x$	$\left\{x \,\middle|\, x \neq k\pi\right\}$	$(-\infty, 1] \cup [1, \infty)$	2π	N/A

The trigonometric functions are periodic functions in that they repeatedly hit the whole range of possible values.

The **amplitude** of a trigonometric function is the height variation. If M is the greatest value of y and m is the smallest value of y then the amplitude is $A = \frac{1}{2}|M - m|$. We can determine the amplitude of a trigonometric function by looking at the coefficient in front of it. For example, $y = A \sin x$ has an amplitude of A. In other words, the graph of $2 \sin x$ is like the graph of $\sin x$ but stretched vertically by a factor of two. If the coefficient is negative, then the graph is reflected about the x axis. For example, $-2 \sin x$ is an upside-down version of $2 \sin x$. Note: The amplitude is always positive.

The **period** of a trigonometric function is the x-axis repetition. It is the smallest value p such that $f(x + p) = f(x)$. The period of the sine and cosine functions is 2π and the period of the tangent function is π. A coefficient on the argument of a trig function, for example the B in $\sin Bx$ affects the period of the graph. For example, $\sin 2x$ has a period half as long as $\sin x$ and $\sin \frac{x}{2}$ has a period twice as long as $\sin x$. In general, for positive B, the period for sine and cosine will be $\frac{2\pi}{B}$.

Example:

What is the period of $y = 2\cos(3x)$?

The typical period of a cosine function is 2π. Put the argument of the function (i.e. $3x$) into the inequality
$$0 \leq 3x \leq 2\pi,$$
and solve for the variable x. Doing so, we find that the period of $y = 2\cos(3x)$ is $\frac{2\pi}{3}$.

In situations where the coefficient of the argument of a trig function is negative, convert it to positive by using knowledge about the even and odd functions. For example,
$$y = \sin(-x) = -\sin x,$$
because sine is an odd function. Also
$$y = \cos(-x) = \cos x,$$
because cosine is an even function.

The **frequency** of a trigonometric function is related to the period via
$$f = \frac{1}{p},$$
and it is measured in Hertz (Hz), which is cycles per second.

Just like other graphs, the trig graphs have zeros wherever there's an x-intercept.

When it comes to horizontal and vertical translations of trig functions, the principles are just the same as with any other function–if the entire function has an added constant it's a vertical translation (up for $+$ and down for $-$) and if the argument of the function has an added constant it's a horizontal translation (left for $+$ and right for $-$).

<div style="border:1px solid">

Example:

Graph the function $f(x) = -4\cos\left(3x + \frac{2\pi}{3}\right) + 3$.

To find the period of the function, we put the argument of the function into an inequality of the starting and ending position of the period of a standard cosine function

$$0 \le 3x + \frac{2\pi}{3} \le 2\pi,$$

and solve for the variable to get

$$-\frac{2\pi}{9} \le x \le \frac{4\pi}{9},$$

which means one period of the function starts at $x = -\frac{2\pi}{9}$ and ends at $x = \frac{4\pi}{9}$. The leading coefficient of four means the graph is stretched vertically by a factor of 4 and the negative sign means the graph is reflected about the x axis. Finally, the 3 added at the end means the whole graph is shifted upwards by 3 units.

Another way to do this is to convert the argument of the function from the form $(Bx + C)$ to the form $(B[x - h])$. If we do so, we get $f(x) = -4\cos\left(3\left[x + \frac{2\pi}{9}\right]\right) + 3$, which indicates a cosine function that is translated $\frac{2\pi}{9}$ units to the left.

</div>

For a trigonometric function with an argument of the form $(Bx + C)$, as in the example given above, the constant C is the **phase** of the function. If the argument is in the form $(B[x + h])$, then the phase is $C = Bh$. The phase is important when dealing with real-life applications of these functions.

If x represents time, then the phase is the fraction of a standard period of 2π that a point on the graph of $y = \sin(Bx+C)$ lags or leads a corresponding point on the graph of $y = \sin Bx$. Horizontal shift is positive for shifts to the right. Phase is opposite in sign from horizontal shift.

For functions of the form $y = \sin(Bx + C)$ or $y = \cos(Bx + C)$

$$\begin{aligned}
\text{Period} &= \frac{2\pi}{B} \\
\text{Horizontal Shift} &= -\frac{C}{B} \\
\text{Phase} &= C
\end{aligned}$$

For functions of the form $y = \sin[B(x - h)]$ or $y = \cos[B(x - h)]$

$$\begin{aligned}
\text{Period} &= \frac{2\pi}{B} \\
\text{Horizontal Shift} &= h \\
\text{Phase} &= -Bh
\end{aligned}$$

When dealing with trig functions other than sine and cosine, the graphing methods will be mostly the same. Keep in mind that the other functions may not have a standard period of 2π so you'll need to slightly revise the inequality process and other methods used to find the period. When graphing the cosecant and secant functions, graph their reciprocals (sine or cosine) first. This makes it easy to tell the important points for the cosecant and secant functions.

Tip:
Always, always, make sure you're operating your calculator in the right mode!

To find the equation from a graph, first determine whether the simplest equation will be the sine or cosine function. Then, build an equation using $f(x) = k + A\sin(B[x - h])$ or use cosine instead of sine. To find B, set the period of the graph equal to $\frac{2\pi}{B}$ and solve for B. A is the amplitude and k and h are the vertical and horizontal transformations.

1.12   Inverse Trig Functions

The inverse of the trig functions are not functions themselves because none of them are one-to-one. The only way that the inverses can be functions is if the domain of the original function is restricted.

The notation

$$y = \sin^{-1} x \text{ or } y = \arcsin x,$$

means $x = \sin y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

To find a value of an inverse function

1. Graph the original function to determine the x-bound that makes it one-to-one
2. Reflect that portion around the line $y = x$
3. Draw a circle thing and mark main angles in radians—cross out the ones that are not in the domain of the original bounded function.
4. Deduce the answer.

Be careful with questions like $\sin^{-1}(\sin(135°))$. They may be expecting the answer $45°$ instead of $135°$. Both are correct but only $45°$ is within the standard bounds of arcsine.

Example:

Evaluate $\tan\left(\cos^{-1}\frac{3}{5}\right)$.

We know that the inside, $\cos^{-1}\frac{3}{5}$, equals some angle such that $\cos\theta = \frac{3}{5}$. Since cosine is adjacent over hypotenuse, we can construct a triangle and label the adjacent leg as 3 and the hypotenuse as 5. Then the angle is θ and using the Pythagorean theorem, we find that the other leg is 4. Taking the tangent of the angle gives us $\tan\theta = \frac{4}{3}$, so

$$\tan\left(\cos^{-1}\frac{3}{5}\right) = \frac{4}{3}.$$

1.13 Proving Identities

Use your calculator to graph both sides to see if they match. This wouldn't be proof but it would be strong evidence.

Many times you have to use an algebraic manipulation such as factoring the difference of squares.

You can also manipulate the side of an equation by multiply it by 1 in any form such as $\frac{1-\sin t}{1-\sin t}$.

To show that an equation is not an identity, prove a counter example by substituting a real number such as π for the variable and solving to reveal a contradiction.

Tip:

- If one side contains multiple fractions, combine them into a single fraction
- Rewrite $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$
- Look for opportunities to make the substitution $\sin^2 x + \cos^2 x = 1$
- If the argument of a trig function contains a sum or difference, then use the relevant sum or difference formula

---

1.14 Trigonometric Formulas

1.14.1 Sum and Difference Formulas

Example:

Find the exact value of $\sin 75°$.

Since $\sin 75° = \sin(45°+30°)$, we can use the sum formula for the sine function

$$
\begin{aligned}
\sin 75° &= \sin(45° + 30°) \\
&= \sin 45° \cos 30° + \cos 45° \sin 30° \\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} \\
&= \frac{1 + \sqrt{3}}{2\sqrt{2}}.
\end{aligned}
$$

$$
\begin{aligned}
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(A - B) &= \sin A \cos B - \cos A \sin B \\
\tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} = \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{aligned}
$$

1.14.2   Double-Angle Formulas

$$
\begin{aligned}
\sin 2A &= 2\sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
\tan 2A &= \frac{2\tan A}{1 - \tan^2 A}
\end{aligned}
$$

1.14.3   Half-Angle Formulas

$$
\begin{aligned}
\sin \frac{A}{2} &= \pm\sqrt{\frac{1 - \cos A}{2}} \\
\cos \frac{A}{2} &= \pm\sqrt{\frac{1 + \cos A}{2}} \\
\tan \frac{A}{2} &= \frac{1 - \cos A}{\sin A}
\end{aligned}
$$

1.14.4   Product to Sum Formulas

$$
\begin{aligned}
\sin A \cos B &= \frac{\sin(A + B) + \sin(A - B)}{2} \\
\cos A \sin B &= \frac{\sin(A + B) - \sin(A - B)}{2} \\
\cos A \cos B &= \frac{\cos(A + B) + \cos(A - B)}{2} \\
\sin A \sin B &= \frac{\cos(A - B) - \cos(A + B)}{2}
\end{aligned}
$$

1.14.5   Sum to Product Formulas

$$
\begin{aligned}
\sin A + \sin B &= 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) \\
\sin A - \sin B &= 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right) \\
\cos A + \cos B &= 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) \\
\cos A - \cos B &= -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)
\end{aligned}
$$

Example:

Show that

$$
\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x} = \sin x + \cos x.
$$

$$
\begin{aligned}
\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x} &= \frac{\cos x}{1 - \frac{\sin x}{\cos x}} + \frac{\sin x}{1 - \frac{\cos x}{\sin x}} \\
&= \frac{\cos x}{\frac{\cos x - \sin x}{\cos x}} + \frac{\sin x}{\frac{\sin x - \cos x}{\sin x}} \\
&= \frac{\cos^2 x}{\cos x - \sin x} + \frac{\sin^2 x}{\sin x - \cos x} \\
&= \frac{\cos^2 x(\sin x - \cos x) + \sin^2 x(\cos x - \sin x)}{(\cos x - \sin x)(\sin x - \cos x)} \\
&= \frac{\cos^2 x \sin x - \cos^3 x + \sin^2 x \cos x - \sin^3 x}{2\sin x \cos x - \cos^2 x - \sin^2 x} \\
&= \frac{\cos^2 x \sin x - \cos x[1 - \sin^2 x] + \sin^2 x \cos x - \sin x[1 - \cos^2 x]}{2\sin x \cos x - 1} \\
&= \frac{2\cos^2 x \sin x - 2\sin^2 x \cos x - \sin x - \cos x}{2\sin x \cos x - 1} \\
&= \frac{2\cos x \sin x[\cos x - \sin x] - [\cos x + \sin x]}{2\sin x \cos x - 1} \\
&= \frac{[\cos x + \sin x][2\cos x \sin x - 1]}{2\sin x \cos x - 1} \\
&= \sin x + \cos x \quad \blacksquare
\end{aligned}
$$

1.15   Solving Trig Equations

To solve trigonometric equations is similar to solving regular algebraic equations. First, isolate the trig function being solved for, then determine the points on the graph of the function for which the equation is true.

You can also solve them by graphing each side on your calculator and finding the intersections. If the right side of the equation is zero, then the "zeros" of the equation will be the solutions.

Once the trig function is isolated, you can also use the inverse function on your calculator.

Note: Trig functions may give more solutions than the algebraic equivalents would do to their being periodic functions. Also remember that solutions must be within the range of the function to be valid.

Just like with algebraic functions, you can use factoring or the quadratic equation to solve trig equations. With trig equations you can also use identities to change the functions.

If you square both sides of an equation be sure to check for extraneous solutions.

Tip:
Check your answers or get insight into where the solutions lie by graphing both sides of the trig equation on your calculator. The solutions will then be the intersections of the two plots. Instead of graphing both sides, you can also move all terms to one side (so the other side is zero) and graph the nonzero side. Then your solutions will be the zeros of equation that you graphed.

Always check your solutions to make sure there are no domain errors.

When working with equations that contain a factor in the argument, care must be taken that all solutions are included. If you just up and solve the equation you may exclude additional solutions. For example, with the equation $\cos 2x = \sqrt{\frac{3}{2}}$

1. Simplify the equation as much as possible without affecting the argument(s)
2. Use the inverse function feature to figure out what degrees the entire argument equals
3. Be sure to add the appropriate value (depends on the functions period) times k to get all possible solutions: $2x = 30° + 360°k$
4. Solve for x: $x = 15° + 180°k$
5. Give above as answer or use integers k (starting with 0) to specify all solutions within a given domain such as 0 to 360 degrees.

Example:

Solve
$$\cos \theta \tan \theta - \cos \theta = 0,$$
for $0 \leq \theta \leq 360°$.

Moving the second term to the right side $\cos \theta \tan \theta = \cos \theta$, we get $\tan \theta = 1$, when we divide both sides by $\cos \theta$. We know that $\tan \theta = 1$ at $\theta = 45°$ and $\theta = 225°$, so there are two solutions. However, since we divided both sides by $\cos \theta$, we also have to find the angles where $\cos \theta$ could be zero, since those angles would also make the original equation true. Since $\cos \theta = 0$ at $\theta = 90°$ and $\theta = 270°$, our complete set of solutions is

$$\theta = 45°, 90°, 225°, 270°.$$

Example:

Solve
$$1 - 4\cos \theta = -2\cos^2 \theta,$$
for $0 \leq \theta \leq 360°$.

Rewriting it in standard quadratic form and making the substitution $u = \cos \theta$, we get the equation

$$2u^2 - 4u + 1 = 0.$$

Using the quadratic formula, we find that $u = \frac{2+\sqrt{2}}{2}$ and $\frac{2-\sqrt{2}}{2}$. Returning to our substitution, we have that $\cos \theta = \frac{2+\sqrt{2}}{2}$ and $\cos \theta = \frac{2-\sqrt{2}}{2}$. The first solution is outside of the range of the cosine function, so it is not a valid solution. Using the arcosine function on the calculator for the second possible solution gives us $\theta = 73.0°$. Remember that this is a reference angle—there are actually two solutions: $73.0°$ and $360° - 73.0° = 287.0°$.

Example:

Solve
$$\sin(3\theta - 45°) = -\frac{\sqrt{3}}{2},$$
for $0 \leq \theta \leq 360°$.

Taking the arcsine of the right side tells us that the angle $3\theta - 45°$ is equal to the reference angle $-60°$. Since $-60°$ corresponds to $300°$ and since this reference angle also indicates that $240°$ satisfies the equation, so we have that $3\theta - 45° = 240°, 300°$. However, we know that this is true for all multiples of $360°$, so $3\theta - 45° = 240° + 360°k, 300° + 360°k$. Adding $45°$ to both sides and dividing both sides by 3 gives us

$$\theta = 95° + 120°k, 115° + 120°k.$$

Since we're only asked for those between 0° and 360°, we find specific angles by plugging in integer values of k to get $\theta = 95°, 115°, 215°, 235°, 335°, 355°$.

1.16   Parametric Equations

Parametric equations show x and y as functions of a third variable.

Parametric equations can be graphed on a calculator by setting the mode to **PAR** instead of **FUNC**.

To **eliminate the parameter** from a pair of parametric equations manipulate the equations to get the sine and cosine functions alone on one side with the parameters then substitute the other side of the equation into a Pythagorean identity such as $\sin^2 \theta + \cos^2 \theta = 1$. You can also use other identities if you're given other functions.

> **Example:**
>
> Eliminate the parameter θ from the set of parametric equations
>
> $$\begin{aligned} x &= 3\sec\theta \\ y &= 4\tan\theta. \end{aligned}$$
>
> Solving each equation for the trigonometric function gives us $\sec\theta = \frac{x}{3}$ and $\tan\theta = \frac{y}{4}$. Making these substitutions into the trigonometric identity $\tan^2\theta + 1 = \sec^2\theta$, gives us
>
> $$\frac{y^2}{16} + 1 = \frac{x^2}{9}.$$

> **Example:**
>
> Eliminate the parameter θ from the set of parametric equations
>
> $$\begin{aligned} x &= 3\sin\theta \\ y &= -2\sin\theta. \end{aligned}$$
>
> Solving both of them for $\sin\theta$ and then equating the two, gives us
>
> $$y = -\frac{2}{3}x.$$

1.17   The Law of Sines and Cosines

The **law of sines** for any (not just right triangles) triangle states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

which also implies that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

With this information, we can solve any triangle given two angles and a side. Working with the law of sines greatly simplifies working with left triangles.

If you're given 2 sides and the angle opposite one of the sides, sometimes multiple triangles can be constructed that fit the information. This is called the **ambiguous case**. When doing problems of this sort, always check to see if multiple triangles can be constructed with the given information.

The **law of cosines** can be used to solve any triangles when two sides and the angle between them are given or when all three sides are given.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

1.18   Solving Triangles

First, draw a generic triangle and label the angles A, B, and C. Then the side opposite of angle A will be side a by convention, and so on.

You need three data points (angles or sides) in order to solve any triangle.

If two internal angles are given, use the fact that the internal angles of triangles sum to 180° to find the missing angle.

If you know an angle and the opposite side, use the law of sines. Watch out for the ambiguous case.

For an SAS triangle, use the law of cosines to find the missing side. Then you can use the law of sines to find

one of the missing angles. Finally, find the third missing angle by using the fact that the internal angles sum to 180°.

For an SSS triangle, use the law of cosines to find one of the missing angles. Then you can find the second missing angle using either the law of cosines or the law of sines. The final missing angle can be found by using the fact that the sum of internal angles is 180°.

> **Tip:**
> Always draw your triangle to confirm that the numbers you got make sense.

The ambiguous case occurs when you're dealing with SSA triangles, and it is illustrated below. Notice that there are two possible values for the angles C and B depending on the values of the given angle A and the given sides a and c. To check for the ambiguous case, recall that a positive value for sine means that sine could be in the first quadrant or the second quadrant.

1. Find the unknown angle B using the law of sines. The result is for the angle in quadrant I
2. Using the angle found above as a reference angle, determine the angle in quadrant II that also satisfies the law of sines
3. With the knowledge that the sum of internal angles cannot be more than 180, add the second angle found above to the known angle A. If the result is less than 180°, then the second angle gives a second triangle that fits the given data
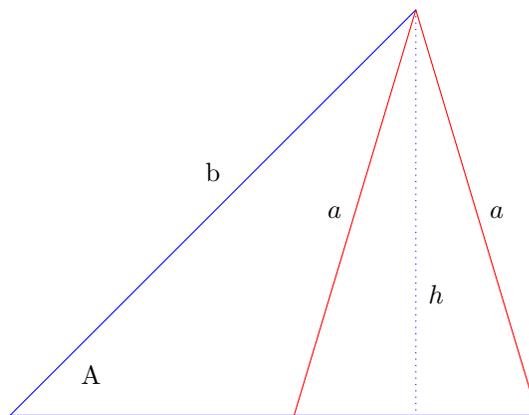
1.19   The Area of a Triangle and Polygons

When you have two sides and the included angle, then the area S of the triangle can be found by taking half of the products of the two sides times the sine of the included angle

$$S = \frac{1}{2}bc\sin A$$
$$S = \frac{1}{2}ac\sin B$$
$$S = \frac{1}{2}ab\sin C.$$

When you have two angles and one side, the area the formulas that can be used are

$$S = \frac{a^2\sin B\sin C}{2\sin A}$$
$$S = \frac{b^2\sin A\sin C}{2\sin B}$$
$$S = \frac{c^2\sin A\sin B}{2\sin C}.$$

To find the area of a triangle when all three sides are known, we use **Heron's formula**

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{1}{2}(a+b+c).$$

Both previous sets of formulas are actually unnecessary because in either case you can use the law of sines or cosines to find all the missing sides. Then all you have to do is use Heron's formula.

The interior angles of a region closed by straight lines should total $(n-2)180°$ where n is the number of sides.

To calculate the area of a polygon, you can usually divide it into a number of triangles and calculate the area of each triangle using one of the above methods.

Another method of calculating the area of polygons uses the coordinates of the vertices of the polygon. This method is sometimes called the **latitude and departure** method.

To use this method with the polygon shown below, label each of the coordinates in order around the polygon.

In our case,

$$
\begin{aligned}
(x_1, y_1) &= (16, 34)\\
(x_2, y_2) &= (14, 15)\\
(x_3, y_3) &= (23, 0)\\
(x_4, y_4) &= (3, 3)\\
(x_5, y_5) &= (0, 16).
\end{aligned}
$$

Then the total area of the 4-sided polygon is given by $A = \frac{1}{2}|(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_5 + x_5, y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_5 + y_5x_1)|$. In our case, $A = \frac{1}{2}|((16)(15) + (14)(0) + (23)(3) + (3)(16) + (0, 34)) - ((34)(14) + (15)(23) + (0)(3) + (3)(0) + (16)(16))| = 139.1.$ [1]

---
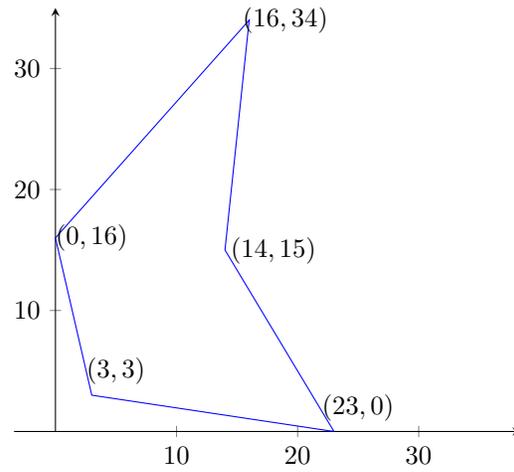
[1] Online Polygon Area Calculator

1.20   Complex Numbers

Remember that division of complex numbers requires using complex conjugates.

The complex numer $x + yi$ can be represented as a vector ending at (x, y) on a 2-dimensional complex plane with a horizontal axis of real numbers and a vertical axis of imaginary numbers. This coordinate system designed for complex numbers is called the **Argand plane**.

The **absolute value** or **modulus** of a complex number is the distance from the origin to the end of the vector. This value is easily found using the Pythagorean theorem.

The **argument** of a complex number denoted as arg(z) is the smallest positive angle theta between the vector and the positive real axis. This angle can be found using basic trigonometry.

This gives us a new method of representing complex numbers

$$x + yi = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta.$$

To convert a complex number from **standard form** to **trigonometric form** use the Pythagorean theorem and a trigonometric function

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}
$$

To convert from trigonometric form to standard form use

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta.
\end{aligned}
$$

These relations can be understood by graphing the the complex number as a point on the Argand plane and then noting that it can be represented by (x, y) and/or by (r, θ).

> **Tip:**
> Always graph the complex numbers you're working with on the Argand plane. You're a lot less likely to make simple mistakes if you do so.

To convert from trigonometric form to standard form on a TI-84 calculator, you can use **P ▶ Rx**(r, θ) from the **ANGLE** button to get the x value of the complex number and use **P ▶ Ry**(r, θ) to get the y value.

Similarly, to convert from standard form to trigonometric form on a TI-84 calculator, you can use **R ▶ Pr**(x, y) from the **ANGLE** button to get the r value of the complex number and use **R ▶ P**$\theta(x, y)$ to get the angle.

To multiply two complex numbers in trigonometric form, multiply their moduli and add their arguments:

$$(r_1 \text{ cis } \theta_1) \cdot (r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2).$$

De Moivre's Theorem is a useful formula for finding roots and powers of complex numbers. It states that

$$
\begin{aligned}
z^n &= r^n \text{ cis } (n\theta) \\
[r(\cos\theta + i\sin\theta)]^n &= r^n(\cos n\theta + i\sin n\theta).
\end{aligned}
$$

Example:

Simplify

$$\left(\sqrt{2} \text{ cis } \frac{7\pi}{24}\right)^8.$$

Using De Moivre's theorem, we get

$$
\begin{aligned}
\left(\sqrt{2} \text{ cis } \frac{7\pi}{24}\right)^8 &= \left(\sqrt{2}\right)^8 \text{ cis } 8 \cdot \frac{7\pi}{24} \\
&= 16 \text{ cis } \frac{7\pi}{3} \\
&= 16 \text{ cis } \frac{\pi}{3}.
\end{aligned}
$$

To divide two complex numbers in trigonometric form, divide their moduli and subtract their arguments

$$\frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2).$$

Every complex number has exactly n distinct nth roots. To find the n, nth roots of a complex number

$$z^{\frac{1}{n}} = \text{ cis } \left(\frac{\theta}{n} + \frac{360°}{n}k\right), \text{ where } k = 0, 1, 2, \ldots n - 1.$$

The root corresponding to $k = 1$ is called the principal nth root of the complex number. Using a calculator will only give the principal nth root of a complex number.

Example:

Find the three cube roots of $32\sqrt{3} + 32i$.

Converting to trig form gives us $32\sqrt{3} + 32i = 64$ cis $30°$. Then using the formula given above, we have that $n = 3$ for cube roots, so the roots are

$$
\begin{aligned}
4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[0]\right) &= 4 \text{ cis } 10° \\
4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[1]\right) &= 4 \text{ cis } 130° \\
4 \text{ cis } \left(\frac{30°}{3} + \frac{360°}{3}[2]\right) &= 4 \text{ cis } 250°
\end{aligned}
$$

Note: The n, nth roots in trigonometric form are distributed evenly (graphically). In other words, the 4, 4th

roots of a complex number will have displayed graphically as 4 vectors that are 90 degrees apart.

1.21   Polar Coordinates

In the Cartesian coordinate system, a point is represented by (x, y). In **polar coordinates** a point is represented by a distance r from the origin and and θ from the positive x-axis, (r, θ). The distance r can be thought of as the length of the vector from the origin to the point. If r is negative then the specified point is r units from the pole in the opposite direction of theta. Unlike for Cartesian coordinates, a point given in polar coordinates is not unique. This is because θ can also be any multiple of $360°$.

In this convention, the origin is called the **pole** and the positive x-axis is called the **polar axis**.

To convert between polar and rectangle coordinates, draw the point using both coordinate systems on the same plane and just use the Pythagorean theorem and/or the following relationships to do the conversions.

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}
$$

To convert between polar and rectangular coordinates using a calculator, use the functions found under the Angle menu.

Equations in polar coordinates use r and θ instead of x and y. To convert between the two, use the same relationships (Pythagorean theorem and/or trig relations) that are used to convert between polar and rectangular coordinates.

> **Example:**
>
> Convert the equation
>
> $$r^2 = 4\sin\theta,$$
>
> written in polar coordinates to one written in Cartesian coordinates.
>
> To convert the equation, we use the relations
>
> $$
> \begin{aligned}
> r^2 &= x^2 + y^2 \\
> y &= r\sin\theta \implies \frac{r}{\sqrt{x^2+y^2}} = \sin\theta,
> \end{aligned}
> $$
>
> to make the substitutions $(x^2 + y^2) = 4\frac{y}{\sqrt{x^2+y^2}}$. Simplifying, we get
>
> $$(x^2 + y^2)^{\frac{3}{2}} = 4y.$$

> **Example:**
>
> The equation
>
> $$x^2 + y^2 = 3x,$$
>
> is written in Cartesian coordinates. Convert it to an equation in polar coordinates.
>
> We use the fact that
>
> $$
> \begin{aligned}
> x &= r\cos\theta \\
> y &= r\sin\theta
> \end{aligned}
> $$
>
> to make the substitutions in the given equation to get $(r\cos\theta)^2 + (r\sin\theta)^2 = 3(r\cos\theta)$. Simplifying gives us
>
> $$r^2 = 3r\cos\theta.$$

To graph a polar equation in a polar coordinate system, just make a table of values for (r, θ) using convenient values of θ such as $30°$, $45°$, $60°$ and so on.

You can graph polar equations on a calculator by setting the function mode to pol.

Another way to graph is to substitute x and y for θ and r, draw that graph, then draw the polar graph by deducing the movement relationships between the two graphs.

In math, there are always multiple ways of solving a problem. Solve it one way and check it the other way. If you can't think of the common method of solving the problem, try to think of alternate ways–perhaps using the advanced functions of your graphing calculator. Don't remember the rules for graphing an equation? You can always just make a table of values or do it on the calculator.

Index