My name is Leon Hostetler. I am currently a student at Florida State University majoring in physics as well as applied and computational mathematics. Feel free to download, print, and use these class notes. If you find them useful, consider buying me a coffee.

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Last Updated: August 25, 2016

Chapter 1

Applied Partial Differential Equations

1.1 Differential Operators

The Gradient

The **gradient** is the generalization fo the derivative to scalar functions of more than one variable. The gradient of a function at a point is a vector pointing in the direction of steepest increase of the function and its magnitude is the slope of the graph at that point.

Recall from elementary calculus that the derivative of a function of a single variable is defined as

$$f'(x) = \lim_{h\to\infty} \frac{f(x+h) - f(x)}{h}.$$

We can also write this simply as

$$\frac{df}{dx} = \frac{f(x+dx) - f(x)}{dx},$$

so the differential of a scalar function of a single variable can be written as

$$df = f(x+dx) - f(x) = \frac{df}{dx}dx.$$

For a function of three variables in cartesian coordinates, we can write

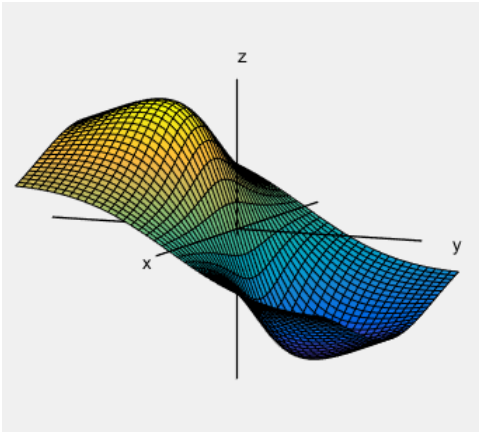
$$\begin{aligned} dU &= U(x+dx, y+dy, z+dz) - U(x,y,z) \\ &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \\ &= \vec{\boldsymbol{\nabla}} U \cdot d\vec{\boldsymbol{r}}, \end{aligned}$$

where $\vec{\boldsymbol{\nabla}}$ is the **gradient** operator and $d\vec{\boldsymbol{r}} = \langle dx, dy, dz\rangle$.

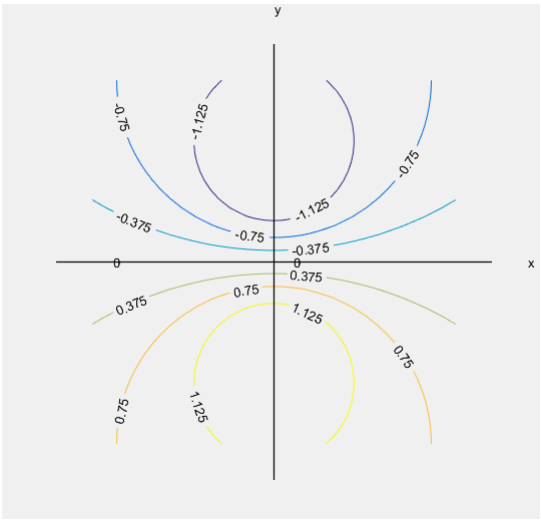
The gradient is an operator that acts on a scalar function and outputs a vector field. In cartesian coordinates, the gradient of a function of three variables $f(x,y,z)$ is

$$\text{grad } f = \vec{\boldsymbol{\nabla}} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{x} + \frac{\partial f}{\partial z}\hat{x}.$$

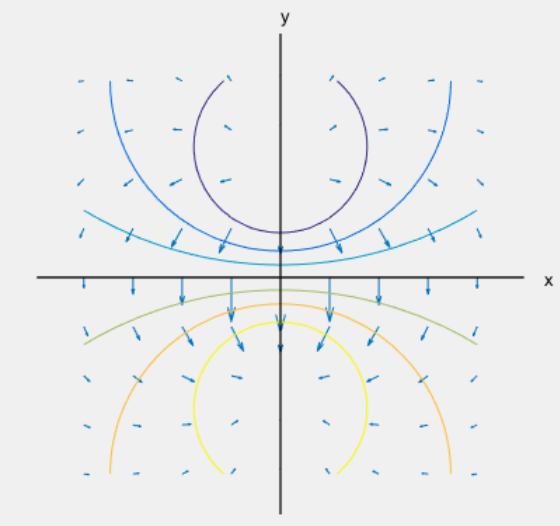
A scalar function of two variables, i.e. a scalar field can be plotted as a surface in three dimensions as the example plot below shows.



Such a plot can also be represented as **level curves** in a two-dimensional plot. The value of the function along any given curve, i.e. the height of the surface above the curve is the same. This is just like any given curve on a topographical map represents the same altitude.



The **gradient** of the scalar field is a vector field which gives the direction and magnitude of the greatest increase of the scalar field at every point. The gradient vector at some point points in the most uphill direction, and its length is proportional to the steepness. Notice that gradient vectors are necessarily perpendicular to the level curve going through those points.



In spherical coordinates, for a function of three variables $f(r, \theta, \phi)$, we know that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial \phi}{\partial \phi}d\phi = \vec{\nabla} \cdot d\vec{r},$$

and that

$$d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi},$$

so the gradient of f is

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \phi}\hat{\phi}.$$

The gradient evaluated at a certain point of some function gives a vector that is orthogonal to the surface formed by the function at that point. This makes it possible to find a normal line to some surface at a given point.

Recall that if you're given a point $\vec{r}_0$ and a vector $\vec{v}$, then the vector equation of the line in the direction $\vec{v}$ and going through the point $\vec{r}_0$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}.$$

You can put this in parametric form by adding the components of the vectors on the right and equating them to the corresponding components on the left. For example, $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$ implies that

$$\begin{cases} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{cases}$$

To find the normal line to a surface at some point $\vec{r}_0$, the gradient of the function evaluated at that gives you the normal vector $\vec{v}$. Then just plug the components into the equations above to get the parametric equations of the tangent line.

To find the tangent hyperplane of a function, use the appropriate partial derivatives to find the slopes in the different directions. For example, if you have a function of three variables $\psi(x, y, z)$, then the tangent hyperplane at a point $P = (x_0, y_0, z_0)$ is given by

$$\psi - \psi_0 = \frac{\partial \psi}{\partial x}\Big|_P (x - x_0) + \frac{\partial \psi}{\partial y}\Big|_P (y - y_0) + \frac{\partial \psi}{\partial z}\Big|_P (z - z_0).$$

The number ψ_0 is simply ψ evaluated at P. This is just a generalization of the equation of the tangent line of a function of one variable

$$y - y_0 = \frac{dy}{dx}\Big|_P (x - x_0).$$

Directional Derivative

The **directional derivative** denoted $\vec{\nabla}_{\vec{v}} f(x_0, y_0, z_0)$ is the rate at which the function $f(x, y, z)$ changes at the point (x_0, y_0, z_0) in the direction of the vector $\vec{v}$. It is calculated as the dot product of the gradient of f with the unit vector in the direction of $\vec{v}$

$$\begin{aligned} \vec{\nabla}_{\vec{v}} f(x_0, y_0, z_0) &= \vec{\nabla} f \cdot \hat{v} \\ &= \vec{\nabla} f \cdot \frac{\vec{v}}{|\vec{v}|}. \end{aligned}$$

If $\hat{v} = \langle v_x, v_y, v_z\rangle$ is the unit vector in the direction $\vec{v}$, then in Cartesian coordinates, the directional derivative of f in the direction $\vec{v}$ is

$$\vec{\nabla}_{\vec{v}} f = \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z.$$

Divergence

The **divergence** of a vector field is the dot product of the gradient operator with the vector field. For a vector field $\vec{V}$,

$$\text{Div } \vec{V} = \vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z.$$

Notice that the divergence is the dot product of two vectors, and is therefore, a scalar field.

To understand divergence, it helps to think of example vector fields and in terms of sources and sinks. Consider a horizontal slice of a room overlaid with a vector field that gives the direction and magnitude of the velocity of the air molecules. In one part of the room, a vacuum is sucking up air and in another part of the room an air tank is blowing air into the room. The vectors near the vaccuum will all be pointing toward the end of the vacuum hose, since all the air in that region is moving

in that direction. Therefore, the location of the vacuum corresponds to a **sink** in the vector field. Near the air tank, on the other hand, the velocity vectors all point away from the nozzle because the higher pressure near the nozzle is forcing the nearby air to move away. The location of the air nozzle corresponds to a **source** in the vector field.

Similarly, you could think of the surface of a pond as a vector field, with the vector field giving the magnitude and direction of the velocity of the water molecules on the surface. This strange pond has a spring underneath one part and an outlet underneath another part. The vectors near the spring all point away from this "source" and the vectors near the outlet all point toward this "sink".

The **divergence** of a vector field at a certain point gives the rate at which *density* exits an infinitesimal region around that point. In other words, if a point is a source in the vector field, then the divergence is positive there, and if the point is a sink, then the divergence is negative there.

Consider a small square on the surface of the aforementioned pond. Note the area of the square as defined by the boundary of the set of water molecules inside the square. In other words, think directly about the particles in that region. As time passes, those particles shift relative to each other if the vector field is not constant. That is, the square becomes distorted as time passes. If the square is in a region of fast flow, it will elongate, and if it's in a region of slow flow it will spread out. If the flow in the region fast and curved, the square will become elongated and curved. If the square is at the source, it will expand as the water molcules are pushed apart by other water molecules coming into existence (so to speak) in that region. Now consider the area of the square before and after a small amount of time has passed. The change in its area is the divergence in that region.

The **Divergence theorem** relates the integral of the divergence of a vector field over a region inside some boundary to the outward flux through the boundary.

$$\int_R \operatorname{div} \vec{\boldsymbol{F}}\, d\vec{\boldsymbol{x}} = \int_{\partial R} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS,$$

where $\vec{\boldsymbol{F}}$ is a vector field and R is a bounded region. The integral on the left is the integral over the region R. The differential $d\vec{\boldsymbol{x}}$ just means $dx\,dy$ if the region is an area and $dx\,dy\,dz$ if the integral is over a volume. The integral on the right is an integral over a boundary denoted by ∂R. If the region under consideration is an area, then the boundary is a curve and the integral is a line integral. If the region is a volume, then the boundary is a surface and the integral is a surface integral. The differential dS is the arc length differential. The vector $\vec{\boldsymbol{n}}$ is the outward pointing normal vector at every point along the boundary.

The integral on the left gives the total area change or volume change of the region. That is, it gives the total "expansion" of the region. The right side gives the total outward flux through the boundary. That is, it gives the amount of stuff crossing the boundary of the region. This relationship makes sense. Since matter is conserved, whatever amount of stuff leaving the region and thereby contracting the region must necessarily be the same amount of stuff that moves out of the region across the boundary. If the region is expanding, the new matter in the region must equal the matter that has crossed the boundary into the region.

The divergence theorem is often used to convert boundary integrals (such as line integrals or surface integrals) into regular integrals over a region and vice versa.

Example:

Show that the divergence of a vector field is invariant under a plane rotation.

For a plane rotation of vector field $\vec{\boldsymbol{V}}$, we have that $\vec{\boldsymbol{V}}' = R\vec{\boldsymbol{V}}$, where R is the orthogonal rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Performing the transformations, we have that

$$\begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix},$$

which gives us

$$\begin{aligned} V_x' &= V_x\cos\theta - V_y\sin\theta \\ V_y' &= V_x\sin\theta + V_y\cos\theta. \end{aligned}$$

Differentiating the above equations gives us

$$\begin{aligned} \frac{\partial V_x'}{\partial x'} &= \frac{\partial V_x}{\partial x'}\cos\theta - \frac{\partial V_y}{\partial x'}\sin\theta \\ \frac{\partial V_y'}{\partial y'} &= \frac{\partial V_x}{\partial y'}\sin\theta + \frac{\partial V_y}{\partial y'}\cos\theta. \end{aligned}$$

Recall the chain rule for partial derivatives. We can expand the derivatives on the right side as

$$\begin{aligned} \frac{\partial V_x}{\partial x'} &= \frac{\partial V_x}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial x'} \\ \frac{\partial V_y}{\partial x'} &= \frac{\partial V_y}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial x'} \\ \frac{\partial V_x}{\partial y'} &= \frac{\partial V_x}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial y'} \\ \frac{\partial V_y}{\partial y'} &= \frac{\partial V_y}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial y'}. \end{aligned}$$

Substituting these in the pair of equations above gives us

$$\begin{aligned} \frac{\partial V_x'}{\partial x'} &= \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial x'}\right)\cos\theta - \left(\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial x'}\right)\sin\theta \\ \frac{\partial V_y'}{\partial y'} &= \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial y'}\right)\sin\theta + \left(\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial y'}\right)\cos\theta. \end{aligned}$$

Now we look at the inverse transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix},$$

which gives us

$$\begin{aligned} x &= x'\cos\theta + y'\sin\theta \\ y &= -x'\sin\theta + y'\cos\theta, \end{aligned}$$

which gives us the partial derivatives

$$\begin{aligned} \frac{\partial x}{\partial x'} &= \cos\theta \\ \frac{\partial x}{\partial y'} &= \sin\theta \\ \frac{\partial y}{\partial x'} &= -\sin\theta \\ \frac{\partial y}{\partial y'} &= \cos\theta. \end{aligned}$$

Substituting these into the pair of equations above gives us

$$\begin{aligned} \frac{\partial V_x'}{\partial x'} &= \frac{\partial V_x}{\partial x}\cos^2\theta - \frac{\partial V_x}{\partial y}\sin\theta\cos\theta - \frac{\partial V_y}{\partial x}\sin\theta\cos\theta + \frac{\partial V_y}{\partial y}\sin^2\theta \\ \frac{\partial V_y'}{\partial y'} &= \frac{\partial V_x}{\partial x}\sin^2\theta + \frac{\partial V_x}{\partial y}\sin\theta\cos\theta + \frac{\partial V_y}{\partial x}\sin\theta\cos\theta + \frac{\partial V_y}{\partial y}\cos^2\theta. \end{aligned}$$

Adding the two equations gives us

$$\begin{aligned} \frac{\partial V_x'}{\partial x'} + \frac{\partial V_y'}{\partial y'} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \\ \text{div}\,\vec{\boldsymbol{V}}' &= \text{div}\,\vec{\boldsymbol{V}}. \end{aligned}$$

Since the divergence of the rotated vector has the exact same form as the divergence of the original vector except with the original components swapped with the new components, this shows that the divergence of a vector field is invariant under plane rotation.

Curl

Using the BAC-CAB identity, we can easily find the curl of the curl of a vector field $\vec{\boldsymbol{F}}$ as

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) &= \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}) - (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{F}} \\ &= \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}) - \vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{F}}.\end{aligned}$$

Example:

Maxwell's equations in a vacuum are

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} &= 0 \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} &= 0 \\ \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} &= \frac{1}{c^2}\frac{\partial \vec{\boldsymbol{E}}}{\partial t} \\ \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} &= -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}.\end{aligned}$$

Use them to derive a wave equation for the electric and magnetic field. Taking the curl of the left side of third equation gives us

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) &= \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}) - \vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{B}} \\ &= -\nabla^2 \vec{\boldsymbol{B}}\end{aligned}$$

Taking the curl of the right side gives us

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times \left(\frac{1}{c^2}\frac{\partial \vec{\boldsymbol{E}}}{\partial t}\right) &= \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} \\ &= \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\right) \\ &= -\frac{1}{c^2}\frac{\partial^2 \vec{\boldsymbol{B}}}{\partial t^2}.\end{aligned}$$

Equating the two gives us

$$\nabla^2 \vec{\boldsymbol{B}} = \frac{1}{c^2}\frac{\partial^2 \vec{\boldsymbol{B}}}{\partial t^2}.$$

Performing the same process on the fourth equation gives us

$$\nabla^2 \vec{\boldsymbol{E}} = \frac{1}{c^2}\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2}.$$

Laplacian

The **Laplacian** is the dot product of the gradient with itself. In other words, it is the **divergence of the gradient** of a function. In 3-dimensional Cartesian coordinates, the Laplacian of f is

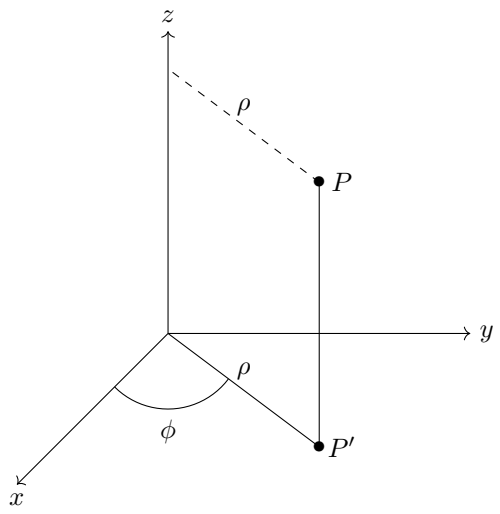
$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{1.1}$$

We use these notations interchangebly. The notation $\nabla^2 f$ is called "del-squared" and is used primarily by physicists, and Δf is called the "Laplacian of f" and is used primarily by mathematicians.

Incidentally, the **divergence** of the gradient of a scalar field f is the same as the Laplacian of f.

$$\operatorname{div}(\vec{\boldsymbol{\nabla}} f) = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} f = \nabla^2 f$$

What is the Laplacian in polar or cylindrical coordinates?



The transformation equations are

$$\begin{aligned}x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z.\end{aligned}$$

In cylindrical coordinates, the function f is a func-

tion of ρ, ϕ, and z, so by the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \\ \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z}.\end{aligned}$$

We could get these partials by inverting the transformation equations, solving them for ρ, ϕ, and z, but it's easier to use the **Jacobian** transformation matrix.

To transform from Cartesian to cylindrical coordinates, we use the Jacobian

$$J\left(\frac{x,y,z}{\rho,\phi,z}\right) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix}.$$

By taking the appropriate partial derivatives of the transformation equations, we get

$$J\left(\frac{x,y,z}{\rho,\phi,z}\right) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now to get the partial derivatives of the inverse transformations, that is, the partial derivatives of the transformation from cylindrical coordinates to cartesian coordinates, we simply invert the matrix above. This can be easily done by treating the top left as a 2×2 matrix and the 1 in the bottom right as a 1×1 matrix, so we don't have to invert the whole 3×3 matrix at once.

$$\begin{aligned} J\left(\frac{x,y,z}{\rho,\phi,z}\right)^{-1} &= J\left(\frac{\rho,\phi,z}{x,y,z}\right) \\ &= \begin{bmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi & 0 \\ \sin\phi & \frac{1}{\rho}\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial \rho}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial \rho}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix}. \end{aligned}$$

Now, we can fill in the partial derivatives of the three equations given above to get

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos\phi\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial y} &= \sin\phi\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\cos\phi\frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial z}.\end{aligned}$$

Next, we have to take the second partial derivatives of these before we can plug them into Eq. (1.1). We start with the first one.

Since f is a function of ρ, ϕ, and z, we use the chain rule to get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\frac{\partial z}{\partial x}.$$

The partial derivative $\frac{\partial z}{\partial x}$ is zero, so the last term is zero. Plugging in $\frac{\partial f}{\partial x}$, we get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial \rho}\left(\cos\phi\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial f}{\partial \phi}\right)\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial f}{\partial \phi}\right)\frac{\partial \phi}{\partial x}.$$

Plugging in $\frac{\partial \rho}{\partial x}$ and $\frac{\partial \phi}{\partial x}$ from the Jacobian gives us

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial \rho}\left(\cos\phi\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial f}{\partial \phi}\right)\cos\phi + \frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial f}{\partial \phi}\right)\left(-\frac{1}{\rho}\sin\phi\right).$$

Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \left[\cos\phi\frac{\partial}{\partial\rho}\left(\frac{\partial f}{\partial\rho}\right) - \sin\phi\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\right)\right]\cos\phi + \left[\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial f}{\partial\rho}\right) - \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right)\right]\left(-\frac{1}{\rho}\sin\phi\right) \\
&= \left[\cos\phi\frac{\partial^2 f}{\partial\rho^2} - \sin\phi\left(-\frac{1}{\rho^2}\frac{\partial f}{\partial\phi} + \frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi}\right)\right]\cos\phi \\
&\qquad + \left[\left(-\sin\phi\frac{\partial f}{\partial\rho} + \cos\phi\frac{\partial^2 f}{\partial\phi\partial\rho}\right) - \frac{1}{\rho}\left(\cos\phi\frac{\partial f}{\partial\phi} + \sin\phi\frac{\partial^2 f}{\partial\phi^2}\right)\right]\left(-\frac{1}{\rho}\sin\phi\right) \\
&= \left[\cos^2\phi\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} - \frac{1}{\rho}\sin\phi\cos\phi\frac{\partial^2 f}{\partial\rho\partial\phi}\right] \\
&\qquad + \left[\frac{1}{\rho}\sin^2\phi\frac{\partial f}{\partial\rho} - \frac{1}{\rho}\sin\phi\cos\phi\frac{\partial^2 f}{\partial\phi\partial\rho} + \frac{1}{\rho^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} + \frac{1}{\rho^2}\sin^2\phi\frac{\partial^2 f}{\partial\phi^2}\right] \\
&= \cos^2\phi\frac{\partial^2 f}{\partial\rho^2} + \frac{2}{\rho^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} - \frac{2}{\rho}\sin\phi\cos\phi\frac{\partial^2 f}{\partial\rho\partial\phi} + \frac{1}{\rho}\sin^2\phi\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\sin^2\phi\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

The second partial derivative with respect to y is very similar, but with the trigonometric functions switched and some of the signs changed.

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\phi\frac{\partial^2 f}{\partial\rho^2} - \frac{2}{\rho^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} + \frac{2}{\rho}\sin\phi\cos\phi\frac{\partial^2 f}{\partial\rho\partial\phi} + \frac{1}{\rho}\cos^2\phi\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\cos^2\phi\frac{\partial^2 f}{\partial\phi^2}.$$

The second partial derivative with respect to z is just

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z^2}.$$

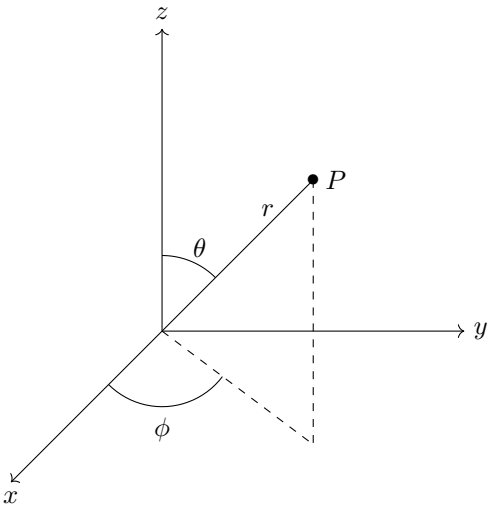
Adding these three second partial derivatives and simplifying gives us the Laplacian for cylindrical coordinates.

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Sometimes, the first two terms are combined like this

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Then when applying it, the product rule has to be used to expand the first term.



In polar coordinates (two-dimensional), $\frac{\partial^2 f}{\partial z^2} = 0$, so the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}.$$

In **spherical coordinates**, the transformations are

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta.
\end{aligned}$$

The Laplacian is calculated in the same manner as it was for cylindrical coordinates, but the process is even more tedious.

Here is the Laplacian for spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

The first two terms have to be unpacked via the product rule.

1.2 Introduction

In this chapter, we focus primarily on three specific partial differential equations (PDEs). Namely, the

1. heat equation,
2. wave equation, and
3. Laplace's equation.

The **heat equation** has the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

in one spatial dimension, and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

in two dimensions. Other notation includes

$$u_t = u_{xx} + u_{yy},$$

and

$$u_t = \nabla^2 u = \Delta u,$$

where Δu is the Laplacian of u.

The **wave equation** has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

in one dimension and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

in two dimensions.

The **Laplace equation** has the form

$$0 = \frac{\partial^2 u}{\partial x^2},$$

in one dimension and

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

in two dimensions.

These three are the most important *linear* PDEs. If a PDE is linear it means that its solutions can be combined, this is called "superposition", by addition and scalar multiplication to get new solutions. In other words, linear combinations of the solutions of linear PDEs are also solutions of those PDEs. Nonlinear PDEs are much more difficult to solve.

These three PDEs are "second order" since the highest derivative that shows up is a second derivative.

The heat equation is sometimes called a **parabolic equation**, the wave equation is sometimes called a **hyperbolic equation**, and the Laplace equation is sometimes called an **elliptic equation**. This is only because these equations bear a vague structural resemblance to their geometric counterparts.

1.3 The Heat Equation

The one-dimensional form of the heat equation describes the motion of heat through a one-dimensional solid. Consider a wire of length L shown below.

To treat this as a one-dimensional problem, we assume that no heat moves through the sides of the wires–only through the ends. That is, the sides of the wires are perfectly insulated.

The **heat energy density** in the wire at position x and time t is denoted by $e(x,t)$. Recall that to calculate the total amount of something in a region, given its density, you integrate the density over that region. So the total **heat energy** in the region between $x = a$ and $x = b$ at time t is given by

$$E_a^b(t) = \int_a^b e(x,t)\,dx.$$

This total heat energy in that region between $x = a$ and $x = b$ will change over time for two reasons:

1. Heat can be added or removed from the system (in this case the wire) as a whole. For example, running a current through the wire or chemical reactions inside the wire could be sources of internal heat. We denote the rate at which heat is being added to the system at position x and time t by $Q(x,t)$. Of course if we're only considering a small region of that wire between $x = a$ and $x = b$, we care only about the heat being added in that region. We can think of Q as being the internal sources of heat.
2. Heat can move around the system via conduction. We denote the **heat flux** in the system at position x and time t by $\phi(x,t)$. By convention, to the right is the positive direction of heat flow.

Tip:
Remember that Q is internal heating. It says nothing about external heating. For example, a wire can have zero internal heating $Q(x,t) = 0$ and still have a temperature gradient and a flux ϕ through the endpoints if the one end is immersed in a hot water bath and the other end is immersed in a cold water bath, for example.

So the change over time of the total heat energy between $x = a$ and $x = b$ is given by

$$\frac{d}{dt}E_a^b(t) = \int_a^b Q(x,t)\,dx + \phi(a,t) - \phi(b,t).$$

The last two terms on the right give the heat flux through the boundary (i.e. end points) of the wire since we only care about the flux at the boundary. Together, the last two equations give us

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \int_a^b Q(x,t)\,dx + \phi(a,t) - \phi(b,t).$$

On the left side, we have the derivative of the integral with respect to time. We can bring this inside the integral as a partial derivative with respect to time since x is being held constant when differentiating with respect to time. Using the fundamental theorem of calculus, we can combine the two flux terms in a single integral.

$$\int_a^b \frac{\partial e}{\partial t}\,dx = \int_a^b Q(x,t)\,dx - \int_a^b \frac{\partial \phi}{\partial x}\,dx.$$

This is the heat equation, also called the integral form of the conservation of heat energy.

We can rearrange the last equation by combining everything in one integral

$$\int_a^b \left(\frac{\partial e}{\partial t} - Q + \frac{\partial \phi}{\partial x}\right) dx = 0.$$

This is true for every choice of a and b provided that $0 \le a \le b \le L$. There is a principle which states that if a function $f(x)$ is defined for all $0 \le x \le L$ and $\int_a^b f(x)\,dx = 0$ for all a and b, then $f(x) = 0$. Therefore,

$$\frac{\partial e}{\partial t} - Q + \frac{\partial \phi}{\partial x} = 0,$$

which gives us the differential form of the conservation of heat energy

$$\frac{\partial e}{\partial t} = Q - \frac{\partial \phi}{\partial x}.$$

This conservation equation is essentially true for any quantity that is conserved. For example, instead of heat in a wire, it could represent pollutant in a canal. As such, the heat equation is also called the **diffusion equation**.

Keep in mind that heat energy is not the same as **temperature**. The two quantities are related by the **heat capacity** or **specific heat** of the material denoted c. The heat capacity is the amount of heat energy that must be added to a unit mass of material to raise its temperature by one degree. For example, since the heat capacity of water is higher than the heat capacity of oil, it takes longer to bring an amount of water to a boil than it would to bring the same amount of oil to a boil. The heat energy and temperature of a material are related as

$$c = \frac{\partial e}{\partial u},$$

where u is the temperature and e is the heat energy. Although the heat capacity, c, for most materials varies slightly with temperature, we'll treat it as a constant.

The relation between heat energy e, the heat capacity c, and the temperature u can also be expressed as

$$e = c\rho u,$$

where ρ is the mass density of the material.

It is approximately true that heat flows from regions of high temperature to regions of low temperature at a rate that is proportional to the temperature gradient. This leads to **Fourier's law**

$$\phi = -K_0 \frac{\partial u}{\partial x}, \tag{1.2}$$

where ϕ is the heat flux, $\frac{\partial u}{\partial x}$ is the temperature gradient in one dimension, and K_0 is a proportionality constant called the **thermal conductivity** of the material. The minus sign is there because the heat flows in the opposite direction of the gradient. Fourier's law is an empircal observation.

Tip:
Notice that the flux has a negative sign, so when given a gradient through the right endpoint of the wire $\frac{\partial u}{\partial x}(L) < 0$, which is negative, this cancels the negative in Fourier's law and indicates a flux out the right end of the wire.

We can now rewrite the differential form of the heat equation

$$\begin{aligned} \frac{\partial e}{\partial t} &= Q - \frac{\partial \phi}{\partial x} \\ \frac{\partial}{\partial t}(c\rho u) &= Q + \frac{\partial}{\partial x}(K_0 \frac{\partial u}{\partial x}). \end{aligned}$$

If we're only considering uniform rods, then the thermal conductivity K_0 does not depend on x, so we can pull that constant outside the partial derivative. Then, dividing both sides by $c\rho$ gives us

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q, \tag{1.3}$$

where $k = \frac{K_0}{c\rho}$ is the **thermal diffusivity**, and $q = \frac{Q}{c\rho}$. The first term on the right is the diffusion term, and the second term on the right gives the sources of heat being

applied to the system (the wire in this case). Generally, we will be given q, and k is just a property of the material, so the only unknown is the temperature $u(x,t)$ with respect to position x and time t. If $q = 0$, then no heating or cooling is being applied to the system, and the heat equation only describes the movement of heat in the system as it equilibrates, that is, as the heat flows from hot regions to cool regions.

To get specific solutions to the heat equation, we need additional information. An initial value problem is a problem in which the function we seek is specified for $t = 0$. In the case of the one dimensional wire problem, we want the solution $u(x,t)$ of the one-dimensional heat equation. If we are given **initial conditions** then we know $u(x,0) = f(x)$. That is, we are given a function that gives the temperature at any position along the wire at time $t = 0$.

Typically, however, we'll be given boundary value problems in which the temperature at the boundary is specified for all time t. These are called the **boundary conditions**. In the case of the one-dimensional wire, the boundaries are the endpoints of the wires, and to be given the boundary conditions means we're given the condition (either temperature or heat flux) at the endpoints for all times t.

There are four different kinds of boundary conditions that we might be given. The first kind are the **Dirichlet boundary conditions**. This is when the temperature at the boundary (the endpoints of the wire in this case) are specifiied for all time. That is, we are given

$$\begin{aligned} u(0,t) &= u_0(t) \\ u(L,t) &= u_L(t), \end{aligned}$$

where the temperature at the left end of the rod $u(0,t)$ is specified as a function of time $u_0(t)$, and the temperature at the right end of the rod $u(L,t)$ is specified as a second function of time $u_L(t)$.

To find specific solutions to the heat equation means to find a function or functions $u(x,t)$ that satisfy the differential form of the heat equation. At first, we will only be interested in **equilibrium solutions** or **steady-state solutions**. If we apply steady forcing, that is, q does not depend on time, and steady bounday conditions, that is, the boundary conditions don't depend on time, then for physical reasons, we can expect the solution to the heat equation to approach some steady state.

To be in equilibrium, it must not depend on time. If the solution does not depend on t, then $\frac{\partial u}{\partial t} = 0$, so our heat equation becomes

$$-k\frac{\partial^2 u}{\partial x^2} = q.$$

Since u depends only on x and not t (since we're only considering equilibrium solutions), we can write this as the ordinary differential equation

$$-k\frac{d^2 u}{dt^2} = q.$$

This is a separable ODE and it can be solved by separating the variables and integrating twice. The two boundary conditions will usually determine the two constants of integration.

Once we have specified the initial conditions, one kind of boundary conditions, the thermal diffusivity k, and the sources of heat $q(x,t)$, then the heat equation Eq. 1.3 has a unique solution.

Tip:

Dirichlet boundary conditions always give unique equilibrium solutions.

Example:

Consider a wire in which $q(x,t) = 0$. This means there are no internal sources of heat such as chemical reactions. All that is happening is that the heat in the bar is flowing from hotter regions to cooler regions.

The temperature at the endpoints are constants,

$$\begin{aligned} u(0,t) &= u_0 \\ u(L,t) &= u_L. \end{aligned}$$

Notice that these are Dirichlet boundary conditions in which the boundary temperatures are constant. Since the endpoints remain at constant temperature, there must be heat flowing into one end of the wire and out the other end.

Find the equilibrium temperature profile of the bar.

Since we are interested in the long-term *equilibrium* solutions, we know that the temperature depends only on position and not on time, so $\frac{\partial u}{\partial t} = 0$ and the heat equation (1.3) becomes

$$\begin{aligned} 0 &= k\frac{\partial^2 u}{\partial x^2} + q \\ \frac{\partial^2 u}{\partial x^2} &= 0. \end{aligned}$$

Since the temperature does not depend on t for an equilibrium solution, this PDE is really the ODE $\frac{d^2u}{dx^2} = 0$. Separating variables and integrating twice gives us

$$u(x) = C_1 x + C_2.$$

Notice that we no longer include t as an independent variable in u since u does not depend on t. This tells us that the equilibrium temperature profile of the wire is a straight line. That is, at any given position, the temperature is constant. There is, however, still heat flowing in the wire if $u_0 \neq u_L$, that is, if there is a temperature difference between the two ends. The temperture at any given point stays constant because the amount of heat flowing out of a region is replaced by more heat flowing into that region.

Example:

Consider the wire again and suppose

$$\begin{aligned} q(x,t) &= \sin\left(\frac{\pi x}{L}\right) \\ u(0,t) &= 0 \\ u(L,t) &= 0. \end{aligned}$$

What is the equilibrium temperature profile of the wire?

If we plot $q(x,t)$, we see that it is 0 at the endpoints of the wire and positive between the endpoints. This means we are adding heat to the middle of the wire. However, the given Dirichlet boundary conditions indicate that we are forcing the temperature of the endpoints to stay a constant 0.

Again, since we are looking for equilibrium solutions, u does not depend on t and $\frac{\partial u}{\partial t} = 0$. In this case, the heat equation becomes

$$\begin{aligned} 0 &= k\frac{d^2u}{dx^2} + q \\ 0 &= k\frac{d^2u}{dx^2} + \sin\left(\frac{\pi x}{L}\right) \\ \frac{d^2u}{dx^2} &= -\frac{1}{k}\sin\left(\frac{\pi x}{L}\right). \end{aligned}$$

This is again a separable ODE and integrating once gives us

$$\frac{du}{dx} = \frac{L}{k\pi}\sin\left(\frac{\pi x}{L}\right) + C_1.$$

Integrating again gives us

$$u(x) = \frac{L^2}{k\pi^2}\sin\left(\frac{\pi x}{L}\right) + C_1 x + C_2.$$

We use the given boundary conditions to find C_1 and C_2. The first one gives us

$$\begin{aligned} u(0,t) &= 0 \\ 0 = u(0) &= \frac{L^2}{k\pi^2}\sin\left(\frac{\pi(0)}{L}\right) + C_1(0) + C_2 \\ C_2 &= 0, \end{aligned}$$

and the second one gives us

$$\begin{aligned} u(L,t) &= 0 \\ 0 = u(L) &= \frac{L^2}{k\pi^2}\sin\left(\frac{\pi(L)}{L}\right) + C_1(L) + C_2 \\ C_1 &= 0. \end{aligned}$$

So our equilibrium solution is

$$u(x) = \frac{L^2}{k\pi^2}\sin\left(\frac{\pi x}{L}\right).$$

The second kind of boundary condition is the **Neumann boundary conditions**. This is when we specify the heat flux at the boundary for all times. That is, we specify the rate at which the heat is moving through the end points.

$$\begin{aligned} -K_0\frac{\partial u}{\partial x}(0,t) &= \phi_0(t) \\ -K_0\frac{\partial u}{\partial x}(L,t) &= \phi_L(t). \end{aligned}$$

Tip:

Neumann boundary conditions do not always give unique equilibrium solutions.

Example:

Find the equilibrium solution for the temperature of the wire if

$$\begin{aligned} q &= 0 \\ \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0. \end{aligned}$$

So we have the situation in which there is no internal heating and the flux at the endpoints is zero. In other words, the endpoints are insulated since there is no heat flowing through the endpoints. Since the flux at the endpoints is specified, these are an example of Neumann boundary conditions.

Since $q = 0$ and u does not depend on time, we again have the case in which the heat equation reduces to

$$\frac{d^2u}{dx^2} = 0.$$

This is a separable equation and integrating twice gives us

$$u(x) = C_1 x + C_2.$$

Differentiating $u(x)$ gives us $\frac{\partial u}{\partial x} = C_1$. Both of the boundary conditions imply $C_1 = 0$, but they say nothing about C_2, so all we know at this point is that

$$u(x) = C_2,$$

is the equilibrium temperature of the wire.

Since none of the heat in the wire is flowing out the endpoints and there are no internal sources of heat, all of the initial heat stays in the wire. That is, the amount of heat energy in the wire is a constant. The equilibrium temperature function $u(x)$ tells us that the entire wire equilibrates to a temperature of C_2 whose value obviously depends on the total heat that was initially in the wire.

For $q = 0$, the heat equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Since the heat energy density is $e = c\rho u(x,t)$, the total amount of heat in the wire is

$$\int_0^L e\,dx = c\rho \int_0^L u(x,t)\,dx.$$

For large t, we know that $u(x,t)$ tends to $u(x) = C_2$ as the temperature in the wire equilibrates. So for large t,

$$c\rho \int_0^L u(x,t)\,dx \approx c\rho \int_0^L C_2\,dx = c\rho L C_2.$$

However, if the initial temperature of the wire (at $t = 0$) is $u(x,0) = f(x)$, and if the total amount of heat in the wire stays the same, then

$$c\rho \int_0^L u(x,0)\,dx = c\rho \int_0^L f(x)\,dx = c\rho L C_2.$$

This tells us that

$$C_2 = \frac{1}{L}\int_0^L f(x)\,dx,$$

that is, the equilibrium temperature C_2 is the average value of the initial temperature function $f(x)$.

Example:

Find the equilibrium profile of the wire if

$$\begin{aligned} q &= 1 \\ \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0. \end{aligned}$$

Since the flux at each endpoint is specified, this is again an example of Neumann boundary conditions.

Plugging q into the heat equation and using the fact that u does not depend on t in the longterm, we get

$$\frac{d^2u}{dx^2} = -\frac{1}{k}.$$

This is a separable ODE and integrating once gives us

$$\frac{du}{dx} = -\frac{1}{k}x + C_1.$$

Integrating again gives us

$$u(x) = -\frac{1}{2k}x^2 + C_1x + C_2.$$

Plugging the boundary conditions into $\frac{du}{dx}$ gives us

$$\begin{aligned} C_1 &= 0 \\ C_1 &= \frac{L}{k}, \end{aligned}$$

and since both cannot be true, there is no equilibrium solution in this case.

We can think of it physically as adding heat to the inside and not allowing any out through the ends, so the wire gets hotter and hotter forever. There is no equilibrium.

The third type of boundary condition is the **Robin boundary conditions**. This is the case in which the boundary satisfies Newton's law of cooling.

$$\begin{aligned} -K_0\frac{\partial u}{\partial x}(0,t) &= -H(u(0,t) - u_0(t)) \\ -K_0\frac{\partial u}{\partial x}(L,t) &= -H(u(L,t) - u_L(t)), \end{aligned}$$

where H is the heat transfer coefficient. These boundary conditions are appropriate when the temperature of the ends of the wire follow Newton's law of cooling. For example, if the hot endpoint of a wire is immersed in a cool water bath, then heat energy will flow out through the endpoint and into the water bath. Newton's law of cooling is valid when the outside is not significantly affected by the inside. In this case, we assume that the hot wire doesn't significantly increase the water bath's temperature. Here, $u(0,t)$ is the temperature of the left end of the wire at time t and $u_0(t)$ is the temperature of the outside near the left endpoint.

Robin boundary conditions are the ones most often encountered in nature, but they make the PDEs more difficult to solve, so we will typically use the easier boundary conditions.

The fourth and final type of boundary condition that we will consider is the **periodic boundary conditions**. This is the case in which the two ends have the same conditions—same temperature and same heat flux. Think of the wire as being bent into a loop so that the left end is in contact with the right end. The temperature and heat flux at both ends is now necessarily the same.

$$\begin{aligned} u(0,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(L,t). \end{aligned}$$

Tip:

Keep in mind when finding equilibrium solutions, that any *equilibrium* solution that you find will generally not fit the initial conditions $u(x,0) = f(x)$. The initial conditions describe the initial temperature of the wire as a function of position. The equilibrium solution describes the temperature of the wire after $t \to \infty$ and the wire has reached an equilibrium temperature with respect to time. A special case occurs when the total heat energy in the wire is conserved then you can integrate the heat energy density using the equilibrium solution as well as the initial conditions and equate the two.

Tip:

To check your work when finding equilibrium solutions, double check your integrations by taking the appropriate partial derivatives of your solution $u(x)$ and plugging them back into the heat equation. Also, be sure to double-check your equilibrium solution against the given boundary conditions.

1.3.1 The Heat Equation in Higher Dimensions

Consider some region U that completely contains a region R. In two dimensions, U is a flat sheet, and in three dimensions, it is a solid. If $e(x,t)$ represents the heat energy density at position $\vec{\boldsymbol{x}} = \langle x, y\rangle$ or $\vec{\boldsymbol{x}} = \langle x, y, z\rangle$ at time t, then the total amount of heat energy in the region R at time t is

$$E_R(t) = \int_R e(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}}.$$

In two dimensions, this integral is

$$E_R(t) = \iint_R e(x, y, t)\, dx\, dy,$$

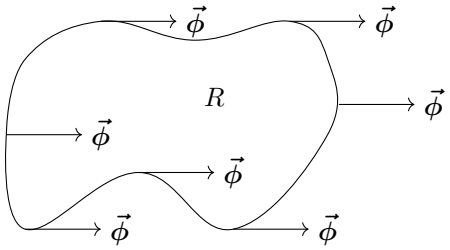
and in three dimensions it is

$$E_R(t) = \iiint_R e(x, y, z, t)\, dx\, dy\, dz.$$

If $Q(\vec{\boldsymbol{x}}, t)$ is the rate at which heat is being added to the system at position $\vec{\boldsymbol{x}}$ and time t, and $\vec{\boldsymbol{\phi}}(\vec{\boldsymbol{x}}, t)$ is the vector field of heat flux at the same position and time, then the rate of change of the total heat energy in R is

$$\begin{aligned} \frac{d}{dt}E_R(t) &= \frac{d}{dt}\int_R e(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} \\ &= \int_R Q(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} - \int_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS. \end{aligned}$$

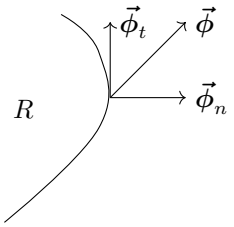
The first integral in the second line is a regular integral over the region or volume R and the second integral is the integral along the boundary of R. The ∂R indicates an integral along the boundary of R. In 2D this is a line integral and in 3D this is a surface integral. The first term gives the amount of heat generated in R, and the second term is the flux term and gives the amount of heat crossing the boundaries. Together, the two give the rate of change of the total heat energy in R.



The vector field $\vec{\boldsymbol{\phi}}(\vec{\boldsymbol{x}}, t)$ gives a direction and flux for the heat flow at every point in R. The arrows point toward cooler areas and away from hotter areas since heat flows away from hot regions. When calculating the change of heat energy in R, we don't care about the flux inside the region, just the flux at the boundaries, since this tells us how much heat energy is entering or leaving R.

However, if the flux is tangent to the boundary as it is in several places in the graphic shown above, there is no actual heat flowing out of R at that point on the boundary. We want to be able to calculate the component of the flux that is actually crossing the boundary.

To do that, we break ϕ into a normal component $\vec{\boldsymbol{\phi}}_n$ and a tangential component $\vec{\boldsymbol{\phi}}_t$ as shown in the graphic below.



The dot product $\vec{\boldsymbol{\phi}}_n \cdot \vec{\boldsymbol{n}}$ where $\vec{\boldsymbol{n}}$ is the vector normal to the boundary, gives the normal component $\vec{\boldsymbol{\phi}}_n$ of $\vec{\boldsymbol{\phi}}$. That's why this quantity appears in the second integral above. If $\vec{\boldsymbol{\phi}}_n \cdot \vec{\boldsymbol{n}}$ is positive, then heat is flowing out of R at that point, and if it is negative, heat is flowing into R at that point.

In 2D, the boundary of R is a curve, so the second integral is a line integral

$$\int_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS = \oint_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS,$$

where dS is the arc length differential. In 3D, the boundary of R is an oriented surface, so it is a surface integral

$$\int_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS = \oiint_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS,$$

where dS is the surface area differential.

In summary, for higher dimensions, we can write the heat equation as

$$\frac{d}{dt}\int_R e(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} = \int_R Q(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} - \int_{\partial R} \vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}}\, dS.$$

In higher dimensions, the relationship between heat and temperature is still $e = c\rho u$ and the relationship between heat flux and temperature is still

$$\vec{\boldsymbol{\phi}} = -K_0 \vec{\boldsymbol{\nabla}} u.$$

The quantity u is the scalar temperature field. The flux vector field $\vec{\boldsymbol{\phi}}$ depends on the gradient of the temperature field.

Rewriting our heat equation using $e = c\rho u$ and the equation for $\vec{\boldsymbol{\phi}}$ gives us

$$\frac{d}{dt}\int_R c\rho\, u(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} = \int_R Q(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} + \int_{\partial R} K_0 \vec{\boldsymbol{\nabla}} u \cdot \vec{\boldsymbol{n}}\, dS.$$

Using the **divergence theorem**, we can rewrite the second integral on the right as an integral over a region R instead of a boundary ∂R.

$$\frac{d}{dt}\int_R c\rho\, u(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} = \int_R Q(\vec{\boldsymbol{x}}, t)\, d\vec{\boldsymbol{x}} + \int_R \operatorname{div}(K_0 \vec{\boldsymbol{\nabla}} u)\, d\vec{\boldsymbol{x}}.$$

Moving the derivative on the first integral inside the integral and moving everything to the same side, we get

$$\int_R \left(c\rho \frac{\partial u}{\partial t} - Q - \operatorname{div}(K_0 \vec{\boldsymbol{\nabla}} u) \right) d\vec{\boldsymbol{x}} = 0.$$

Since this is true for every region R contained in U, it must be the case that the integrand is zero. That is,

$$c\rho \frac{\partial u}{\partial t} = Q + K_0 \operatorname{div}(\vec{\boldsymbol{\nabla}} u).$$

The divergence of the gradient is the **Laplacian**, that is, $\operatorname{div}(\vec{\boldsymbol{\nabla}} u) = \nabla^2 u$, so we can write

$$\begin{aligned} c\rho \frac{\partial u}{\partial t} &= Q + K_0 \nabla^2 u \\ \frac{\partial u}{\partial t} &= \frac{Q}{c\rho} + \frac{K_0}{c\rho} \nabla^2 u \\ \frac{\partial u}{\partial t} &= q + k\nabla^2 u. \end{aligned}$$

This is the **heat equation** for higher dimensions.

Equilibrium solutions are ones that don't depend on time, then the heat equation simplifies to

$$\nabla^2 u = -\frac{q}{k},$$

and in two dimensions, this can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{q}{k}.$$

This last equation is called **Poisson's equation**.

If $q = 0$ the heat equation simplifies even further to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is called **Laplace's equation**.

What does the Laplacian mean in this case? If a point is warmer (i.e. has higher u) than the average of its neighborhood, the Laplacian at that point is negative. This is similar to the fact that if a point on a curve f is lower than the average of its neighbors, then the second derivative of f at that point is greater than zero. That is, the point is at a local minimum. If you're warmer than all your neighbors, you will cool off over time, the Laplacian $\nabla^2 u$ is negative and $\frac{\partial u}{\partial t} < 0$.

What about the boundary conditions in two or three dimensions? Earlier, we were only considering a single dimension.

Dirichlet boundary conditions: The temperature is specified along the boundary ∂U of the region U. That is,

$$u(\vec{\boldsymbol{x}}, t) = g(\vec{\boldsymbol{x}}, t) \quad \text{for } \vec{\boldsymbol{x}} \in \partial U,$$

where g is given.

Neumann boundary conditions: The normal component of the heat flux across the boundary is specified, that is, $\vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}} = -K_0 \vec{\boldsymbol{\nabla}} u$ is specified. Note, a notation commonly used is

$$\frac{\partial u}{\partial \vec{\boldsymbol{n}}} = \vec{\boldsymbol{\nabla}} u \cdot \vec{\boldsymbol{n}}.$$

Robin boundary conditions: Relates the flux through the boundary with the temperature at the boundary using Newton's law of cooling. That is

$$-K_0 \frac{\partial u}{\partial \vec{\boldsymbol{n}}} = H(u(\vec{\boldsymbol{x}}, t) - u_0(\vec{\boldsymbol{x}}, t)),$$

where the left hand side is the heat flux, u_0 is the external temperature, and H is the constant of proportionality.

Periodic boundary conditions: In order for the boundary conditions to be periodic, the region must be a rectangle in two dimensions and a rectangular prism in three dimensions. This is so that the opposite sides of the region are the same. In the two dimensional case of a rectangle of height H and width L, the left and right side have the same temperature and flux

$$\begin{aligned} u(0, y) &= u(L, y) \\ \frac{\partial u}{\partial x}(0, y) &= \frac{\partial u}{\partial x}(L, y), \end{aligned}$$

and the top and bottom have the same temperature and flux

$$\begin{aligned} u(x, 0) &= u(x, H) \\ \frac{\partial u}{\partial y}(x, 0) &= \frac{\partial u}{\partial y}(x, H). \end{aligned}$$

1.4 Separation of Variables

Separation of variables is a solution technique that at its most basic, works for PDEs that are linear and homogeneous.

An **operator**, L, is something that when applied to a function yields another function. Like a function takes in a number and yields a number, and operator takes in a function and yields a function. Examples of operators, L, operating on a function f include

$$\begin{aligned} L = \frac{\partial}{\partial x}; \qquad & Lf = \frac{\partial f}{\partial x} \\ L = 2; \qquad & Lf = 2f \\ L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}; \qquad & Lf = \frac{\partial f}{\partial t} - k\frac{\partial^2 f}{\partial x^2} \\ L(u) = u^2; \qquad & L(\sin x) = \sin^2 x. \end{aligned}$$

Notice that the third operator in the example above gives the heat equation when applied to a function.

An operator, L, is **linear** if

$$L(c_1 f + c_2 f) = c_1 L f_1 + c_2 L f_2.$$

Notice that the first three operators in the list above are linear operators, but the last one is not. The linear operators are "multiplication-like" in that applying the operator to a function is like multiplying the operator by the function. Nonlinear operators are not multiplication-like.

A PDE is **linear** if if it can be written in the form $Lu = f$ where L is some linear differential operator.

A **homogeneous** PDE is one of the form $Lu = 0$. The heat equation is homogeneous if $q(x,t) = 0$. Boundary conditions are homogeneous if they are satisfied by the zero function. For example, the Dirichlet boundary conditions give $u(0,t) = u_0(t)$ and $u(L,t) - u_L(t)$. These are homogeneous if $u_0 = u_L = 0$, so that $u(0,t) = 0$ and $u(L,t) = 0$. So the homogeneous heat equation is $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with homogeneous boundary conditions.

We will illustrate the separation of variables technique by solving the one-dimensional homogeneous heat equation first with Dirichlet boundary conditions, then with Neumann boundary conditions, and finally, with periodic boundary conditions. In all cases, separation of variables is a five step process.

1.4.1 Dirichlet Boundary Conditions

Here we will solve the one-dimensional homogeneous heat equation with Dirichlet boundary conditions. That is, we will solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).$$

Notice that $u(x,t) = 0$ is a trivial solution of the PDE, but we want non-trivial solutions.

Step 1

Separate the variables in the PDE. This step depends only on the PDE and not on the boundary or initial conditions. We will look for solutions of the form

$$u(x,t) = X(x) \cdot T(t).$$

There may be many solutions that will satisfy the PDE, but won't be of this separable form. Taking the partial derivative of this with respect to time and the second partial derivative with respect to x, we can plug this back into the PDE to get

$$X(x) \cdot T'(t) = kX''(x) \cdot T(t).$$

Separating the variables using some simple algebra, we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}.$$

The left side depends only on x and its second derivative, and the right side depends only on t and its first derivative. The only way these two sides can be equal is if they're both constant. For convenience, we'll let this constant be $-\lambda$, so

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda.$$

Now, we can look at each side separately. What started off as a PDE has now become a pair of ODEs linked by the constant λ.

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + \lambda k T(t) &= 0. \end{aligned}$$

When separating variables, you want as few parameters as possible on the eigenvalue equation, which in this and many cases, is the x equation. You definitely don't want a parameter on the highest derivative of the eigenvalue equation. The eigenvalue equation is typically the most difficult of the two equations to solve, and it is where the most work is. It doesn't matter so much if there are parameters on the other ODE as it is in this case with the k, since the non-eigenvalue equation is a lot less work.

Step 2

In the next step, we solve the two ODEs. First, we solve the one containing x, the one often called the **eigenvalue equation**. Here we start using the boundary conditions.

For the PDE to satisfy the boundary conditions, $X''(x) + \lambda X(x) = 0$ has to satisfy the boundary conditions. That is, $X(0) = X(L) = 0$. This is simply a

consequence of defining $u(x,t) = X(x) \cdot T(t)$ and realizing that $X(x)$ is the only part of the right side containing x. So, we want to find non-trivial solutions $X(x)$ to

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0.$$

The non-trivial functions $X(x)$ that satisfy the ODE are called **eigenfunctions**, and the corresponding λ are called **eigenvalues**.

This is a second order constant coefficient ODE. To solve them, we look for exponential solutions of the form

$$X(x) = e^{rx}.$$

Plugging this into the ODE gives us

$$r^2 e^{rx} + \lambda e^{rx} = 0.$$

Since e^{rx} is never 0, we can factor it from the equation, and we get

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \\ &= \pm i\sqrt{\lambda}. \end{aligned}$$

We expect two linearly independent solutions, so our general solution is

$$X(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}.$$

Case 1: If λ is positive, that is $\lambda = \beta^2 > 0$, the general solution is more conveniently written as

$$X(x) = C_1 \cos\beta x + C_2 \sin\beta x.$$

To be an eigenfunction, $X(x)$ must satisfy the boundary conditions, so we now check the boundary conditions. When $x = 0$, $X(x)$ must be zero, which implies that C_1 is zero. Second, when $x = L$, $X(x)$ must be zero, which implies that $C_2 \sin\beta L = 0$. If $C_2 = 0$ then the whole function is zero since the cosine term is already zero. That just gives us the trivial solution, and we want the non-trivial solution, so we need $\sin\beta L = 0$. For this to be the case, $\beta L = n\pi$, where $n = 1, 2, 3, \ldots$. Therefore, $\lambda = \left(\frac{n\pi}{L}\right)^2$ and our solution simplifies to

$$X(x) = C \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

Case 2: If $\lambda = 0$, the ODE becomes $X''(x) = 0$. To solve this, just integrate twice to get

$$X(x) = C_1 + C_2 x.$$

Checking the boundary conditions, we find that $0 = X(0) = C_1$ implies that $C_1 = 0$ and $0 = X(L) = C_2 L$ implies that $C_2 = 0$. But $X(x) = 0$ is just the trivial solution, so $\lambda = 0$ is not an eigenvalue.

Case 3: If λ is negative, that is, $\lambda = -\gamma^2 < 0$, the general solution is more conveniently written as

$$X(x) = C_1 \cosh\gamma x + C_2 \sinh\gamma x.$$

Checking the boundary conditions, we find that $0 = X(0) = C_1$ implies that $C_1 = 0$ and $0 = X(L) = C_2 \sinh\gamma L$ implies that $C_2 = 0$ since $\sinh\gamma x$ is zero only at $x = 0$. So when λ is negative, there are no non-trivial solutions to the ODE, which tells us that there are no negative eigenvalues.

In principle, we should also check for complex eigenvalues. In this particular type of problem, all the eigenvalues are real.

So to summarize Step 2, we found that the solutions to

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0,$$

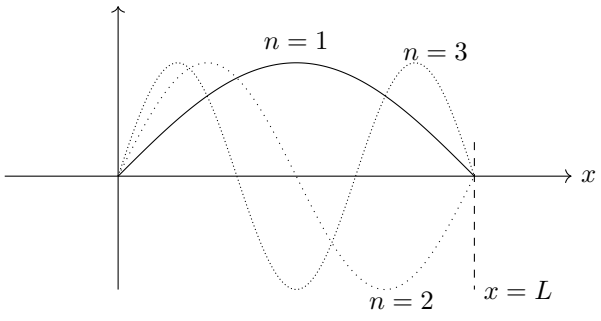
are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \\ X_n(x) &= \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots \end{aligned}$$

It took a lot of time to solve this ODE, so this solution should be memorized.

Notice that we neglect the constant that should appear in this differential equation. This is because an eigenfunction is understood to have multiplicative constant(s), and most people let the constant be 1. Besides, the constants are taken care of later in step 4 of this process where the general solution is written.

An important feature of this solution, since the boundary conditions specify that the function must be zero at the endpoints, is that the function consists of discrete oscillations in the window between $x = 0$ and $x = L$. For $n = 1$, there is a single peak, for $n = 2$ there are two peaks, and so on as shown in the graph below.



Step 3

In step 3, we solve the second ODE, the one containing t. We already know the possible values of λ, and we only use those found in step 2. The ODE

$$T'(t) + k\lambda T(t) = 0,$$

has the solution

$$T(t) = e^{-k\lambda t} = e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Step 4

In step 4, we combine the answers from step 2 and step 3. Remember, in step 1, we started looking for solutions of the form $u(x,t) = X(x) \cdot T(t)$. Multiplying the two ODE solutions we found gives us

$$X_n(x) \cdot T_n(x) = \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

These are *all* of the separated solutions. Our original problem was linear, so linear combinations of this solution are also solutions. That is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

provided the series converges. It turns out that *every* solution can be written in this form. That is, every homogeneous linear PDE can be written as a sum of separable solutions.

Step 5

Use the initial conditions to figure out what the constants B_n are. Recall that the initial conditions are $u(x,t) = f(x)$. Plugging $t = 0$ into the equation above gives us

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Notice that this is the Fourier sine series for $f(x)$.

Next, we multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$, then integrate from 0 to L.

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

Next we switch the order of integration and summation and pull the B_n coefficients outside the integral since they are just constants.

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

To evaluate the second integral, we use the trigonometric identities $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A \sin B$. Subtracting the first from the second gives us $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$. By letting $A = \frac{n\pi x}{L}$ and $B = \frac{m\pi x}{L}$, we can rewrite the second integral as

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \frac{1}{2}\left[\cos\left(\frac{[n-m]\pi x}{L}\right) - \cos\left(\frac{[n+m]\pi x}{L}\right)\right] dx.$$

Now we can integrate the two terms separately since we know how to integrate cosines.

$$\int_0^L \frac{1}{2}\left[\cos\left(\frac{[n-m]\pi x}{L}\right) - \cos\left(\frac{[n+m]\pi x}{L}\right)\right] dx = \frac{L}{2\pi(n-m)} \sin\left(\frac{[n-m]\pi x}{L}\right)\Big|_0^L - \frac{L}{2\pi(n+m)} \sin\left(\frac{[n+m]\pi x}{L}\right)\Big|_0^L.$$

If $n \neq m$ this simplifies to 0. If $n = m$, we can't use this because there would be division by zero, so we have to treat it as the special case

$$\begin{aligned} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx &= \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right] dx \\ &= \frac{1}{2}x\Big|_0^L - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right)\Big|_0^L \\ &= \frac{L}{2}. \end{aligned}$$

So we have that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}$$

So the equation we were looking at before, simplifies to

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \frac{L}{2}.$$

Solving for B_m and replacing m with n, we find that our

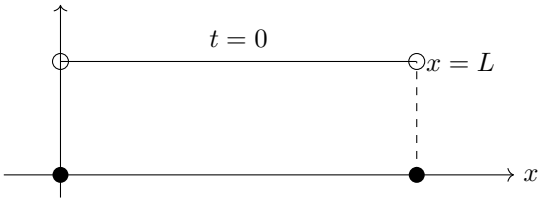
coefficients are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Example Problem

Consider the homogeneous heat equation with Dirichlet boundary conditions and initial conditions $f(x) = 1$. This means, the bar of length L initially has a temperature of 1 all along the bar, except exactly at the endpoints, where the temperature is zero. Since $q = 0$, no heat is being added to the bar and the heat already in the bar will immediately start flowing out the ends.

At $t = 0$, the temperature profile of the bar is



We know that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Plugging $f(x) = 1$ into the formula for B_n, we get

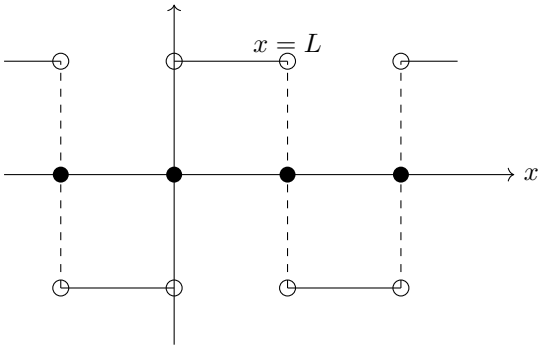
$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx \\ &= -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= -\frac{2}{n\pi}((-1)^n - 1) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

That is, the Fourier sine series for $f(x) = 1$ is

$$1 = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{L}\right).$$

But what happens when $x = 0$. The Fourier series gives us the contradiction $1 = 0$. However, remember that the boundary conditions are that the endpoints are 0, so at $x = 0$, the temperature is 0 rather than 1. That is, the Fourier series given above for $f(x) = 1$ is valid only for $0 < x < L$, and not at $x = 0$ or $x = L$.

If we plot the Fourier series, we get a periodic function that looks like



In our case, we are only interested in the segment $[0, L]$.

So our solution is

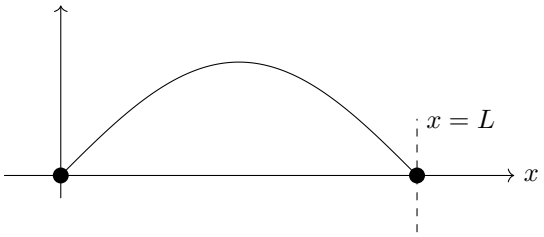
$$u(x,t) = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

The exponential makes this series converge very fast. In fact, just the first term provides a very good approximation when $t \geq \frac{1}{k(\pi/L)^2}$

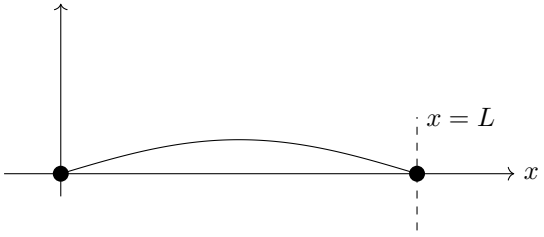
$$u(x,t) \approx \frac{4}{\pi} e^{-k\left(\frac{\pi}{L}\right)^2 t}\sin\left(\frac{\pi x}{L}\right).$$

So the temperature profile curve is a short segment of the sine function that is shrinking downward exponentially quickly.

Above, we showed the temperature profile of the bar at $t = 0$. Shortly thereafter, the temperature profile looks like:



Shortly after that, the temperature profile looks like:



Tip:

When calculating the Fourier coefficients, if $f(x)$ is a polynomial, you must use integration by parts. If it is an exponential, you must do two integrations by parts, rearrange and then cancel stuff, or use an integral table. If $f(x)$ is a sine or cosine, use trigonometric identities. If $f(x)$ is piecewise defined, you must break the integrals into the corresponding pieces.

1.4.2 Orthogonality

Two vectors $\vec{A} = \langle A_1, A_2, A_3 \rangle$ and $\vec{B} = \langle B_1, B_2, B_3 \rangle$ are orthogonal if and only if

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = 0.$$

Two functions $f(x)$ and $g(x)$ are defined as orthogonal over $[0, L]$ if and only if

$$\int_0^L f(x) g(x)\, dx = 0.$$

Notice that these are actually similar definitions since an integral is very much like addition.

When working with Fourier series, the sine and cosine terms are the components of our "vector" space.

Recall that given a vector $\vec{A}$ and a vector $\vec{B}$, we can write $\vec{A}$ as the sum of a vector in the direction of $\vec{B}$ and a vector orthogonal to $\vec{B}$. That is,

$$\vec{A} = A_b \vec{B} + A_p \vec{B}^{\perp}.$$

Taking the dot product of both sides with $\vec{B}$ gives us the coefficient A_b of the projection of $\vec{A}$ in the direction of $\vec{B}$

$$A_b = \frac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}}.$$

This is essentially the same thing we do when finding the Fourier coefficients

$$B_n = \frac{\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx}{\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}.$$

That is, B_n is the coefficient of the projection of $f(x)$ in the direction $\sin\left(\frac{m\pi x}{L}\right)$.

1.4.3 Neumann Boundary Conditions

Here we solve the homogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with Neumann boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

Step 1

In the first step, we separate the variables in order to get a pair of ordinary differential equations. Since the PDE has not changed from the example with Dirichlet boundary conditions, our result for this step is the same

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + \lambda k T(t) &= 0. \end{aligned}$$

Step 2

The second step is to solve the eigenvalue equation $X''(x) + \lambda X(x) = 0$. Since this involves the boundary conditions, this step changes from the example with Dirichlet boundary conditions. We again have to check the different possible cases for the eigenvalue.

Case 1: The eigenvalue is positive, $\lambda = \beta^2 > 0$. The general solution to the eigenvalue equation is

$$X(x) = C_1 \cos \beta x + C_2 \sin \beta x.$$

Next, we use the boundary conditions to find C_1 and C_2. Differentiating the eigenvalue equation, we get

$$X'(x) = -C_1 \beta \sin \beta x + C_2 \beta \cos \beta x.$$

From the first boundary condition, we get

$$X'(0) = C_2 \beta = 0,$$

which implies $C_2 = 0$. From the second boundary condition, we get

$$X'(L) = -C_1 \beta \sin \beta L = 0,$$

which implies $\sin \beta L = 0$, which gives us $\beta L = n\pi$ for $n = 1, 2, 3, \ldots$. This gives us the eigenvalues

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots,$$

for $\lambda > 0$ and the solutions to the eigenvalue equation

$$X_n = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots.$$

Case 2: The eigenvalue is zero, $\lambda = 0$. In this case, the eigenvalue equation simplifies to $X'' = 0$, the general solution which is

$$X(x) = C_1 + C_2 x,$$

is easily found by integrating twice. Differentiating gives us $X'(x) = C_2$. Both boundary conditions imply that $C_2 = 0$, but they say nothing about C_1. Therefore, C_1 is arbitrary, and $\lambda = 0$ *is* an eigenvalue since it gives a nonzero solution $X(x) = C_1$ to the eigenvalue equation with the given boundary conditions. The associated eigenfunction is just $X(x) = 1$.

Case 3: The eigenvalue is negative, $\lambda = -\gamma^2$. The general solution to the eigenvalue equation is

$$X(x) = C_1 \cosh \gamma x + C_2 \sinh \gamma x.$$

Differentiating gives us

$$X'(x) = C_1\gamma \sinh \gamma x + C_2\gamma \cosh \gamma x.$$

From the first boundary condition, we get $X'(0) = C_2\gamma = 0$, which implies $C_2 = 0$. From the second boundary condition, we get $X'(L) = C_1\gamma \sinh \gamma L = 0$, which implies $C_1 = 0$ since $\sinh \gamma L$ cannot be zero. So all solutions here are trivial, that is, there are no negative eigenvalues.

Overall, our eigenvalues are given by

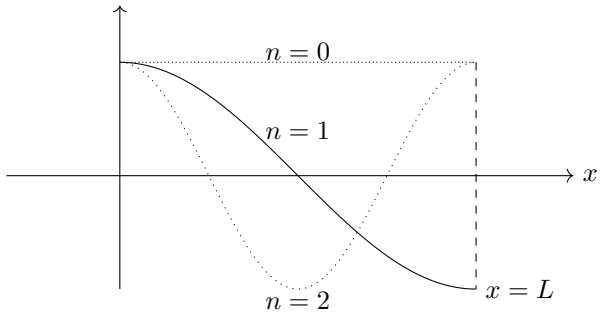
$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, \ldots.$$

Notice that we have included $n = 0$ to take care of the zero eigenvalue. The solutions to the eigenvalue equation are

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 0, 1, 2, 3, \ldots.$$

The key differences between these results and those for Dirichlet boundary conditions is that $n = 0$ is included, and the function is now a cosine instead of a sine.

For Neumann boundary conditions, the derivative needs to be zero at the endpoints, so example solution curves are:



Step 3

In step 3, we solve the time equation, $T'(t) + \lambda k T(t) = 0$. The general solution is $T = e^{-k\lambda t}$. Plugging in the eigenvalues, which are known now, the solution is

$$T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad n = 0, 1, 2, 3, \ldots$$

In particular,

$$T_0(t) = 1.$$

Step 4

In step four, we write the general solution to the PDE as a linear combination of the solution to the eigenvalue equation times the solution to the time equation.

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

Since both functions in the sum return 1 when $n = 0$, the $n = 0$ case is typically separated from the others, and the general solution is written as

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

Step 5

In step 5, we use the initial conditions to find the constants A_0 and A_n. Since $t = 0$ initially, we have that

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right). \tag{1.4}$$

This is the Fourier *cosine* series for $f(x)$. Again, we use the orthogonality of the eigenfunctions to find A_n. To find A_0, we just have to integrate both sides of Eq. (1.4) to get

$$\begin{aligned}\int_0^L f(x)\,dx &= \int_0^L A_0\,dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \\ &= A_0 x\Big|_0^L + \sum_{n=1}^{\infty} A_n \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= A_0 L.\end{aligned}$$

So

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

is just the average value of $f(x)$.

To find the coefficients A_n, we start by multiplying both sides of Eq. (1.4) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating, then switching integration and summation on the right. Using trigonometric identities as we did with Dirichlet boundary conditions, we find that the integral on the right simplifies to

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0, \end{cases}$$

which gives us

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx = 0 + A_m \frac{L}{2},$$

or

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx.$$

We can now write the full solution to the heat equation with Neumann boundary conditions as

$$\begin{aligned} u(x,t) &= A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) \\ A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

Example Problem

Solve the homogeneous heat equation with Neumann boundary conditions and initial conditions $f(x) = u(x,0) = x$. We know the solution is

$$\begin{aligned} u(x,t) &= A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) \\ A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \end{aligned}$$

so we just have to compute A_0 and A_n, that is, compute the Fourier cosine series of $f(x) = x$. Computing A_0, we find

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L x\,dx \\ &= \frac{1}{2L}x^2\Big|_0^L \\ &= \frac{L}{2}. \end{aligned}$$

To compute A_n, we use integration by parts and find

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]. \end{aligned}$$

So the Fourier cosine series of $f(x) = x$ is

$$\begin{aligned} x &= \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{L}{2} - \sum_{\substack{n=1 \\ odd}}^{\infty}\frac{4L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right). \end{aligned}$$

The full solution is then

$$u(x,t) = \frac{L}{2} - \sum_{\substack{n=1 \\ odd}}^{\infty}\frac{4L}{n^2\pi^2}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right).$$

Notice that as $t \to \infty$, $u(x,t) \to \frac{L}{2}$. This is the equilibrium solution, and it shows why, via the Fourier series of the initial conditions, the equilibrium solution involves the average value of the initial conditions.

1.4.4 Periodic Boundary Conditions

Here we solve the homogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with periodic boundary conditions

$$\begin{aligned} u(-L,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) \\ u(x,0) &= f(x). \end{aligned}$$

Notice that the length of the bar is now $2L$.

Step 1

Since the PDE has not changed from the example with Dirichlet or Neumann boundary conditions, our result for this step is the same

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + \lambda k T(t) &= 0. \end{aligned}$$

Step 2

In this step, we solve the eigenvalue problem $X''(x) + \lambda X(x) = 0$ with the given boundary conditions by again considering the three possible cases for λ.

Case 1: The eigenvalue is positive, $\lambda = \beta^2 > 0$. The general solution to the ODE is

$$X(x) = C_1\cos\beta x + C_2\sin\beta x.$$

The boundary condition $X(-L) = X(L)$ implies that $C_2\sin\beta L = 0$. Differentiating $X(s)$ gives us

$$X'(x) = -C_1\beta\sin\beta x + C_2\beta\cos\beta x.$$

The boundary condition $X'(-L) = X'(L)$ implies that $C_1\sin\beta L = 0$. So the boundary conditions tell us that

$$\begin{aligned} C_2\sin\beta L &= 0 \\ C_1\sin\beta L &= 0. \end{aligned}$$

If $\sin\beta L \neq 0$, then it must be that $C_1 = C_2 = 0$. But that gives us $X(x) = 0$, which is the trivial solution, which we don't want. Therefore, $\sin\beta L = 0$, which gives us $\beta L = n\pi$ for $n = 1, 2, 3, \ldots$ When $\sin\beta L = 0$, both C_1 and C_2 are arbitrary, and therefore, both $\sin\beta L$ and $\cos\beta L$ are eigenfunctions.

So the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots,$$

and the eigenfunctions are

$$X(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right),$$

and we say the eigenvalues are "double eigenvalues".

Case 2: The eigenvalues are zero, $\lambda = 0$. In this case, the eigenvalue equation simplifies to $X'' = 0$, which can be solved by integrating twice to get $X(x) = C_1 +$

C_2x. The boundary condition $X(-L) = X(L)$ implies that $C_2 = 0$. The boundary condition $X'(-L) = X'(L)$ implies that $C_2 = C_2$ and says nothing about C_1. Therefore, $C_2 = 0$ and C_1 is an arbitrary constant. This tells us that $\lambda = 0$ is an eigenvalue and the associated eigenfunction is $X(x) = 1$.

Case 3: There are no negative eigenvalues, just as with the previous two examples.

Notice that the eigenvalues and the eigenfunctions for periodic boundary conditions are just the combined eigenvalues and eigenfunctions of the Dirichlet and Neumann boundary conditions. It works out this way because we choose $[-L, L]$ instead of $[0, L]$.

Overall, our eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 1, 2, 2, 3, 3, \ldots,$$

and the eigenfunctions are

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right), \quad X_0(x) = 1.$$

Step 3

In this step, we solve the time equation $T'(t)+\lambda kT(t) = 0$. The solution is the same as before and we just plug the possible eigenvalues into the solution

$$T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad n = 0, 1, 2, 3, \ldots$$

Step 4

In step four, we combine the eigenvalue and time solutions to get

$$\begin{aligned} u(x,t) &= A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Step 5

In step 5, we use the initial conditions $f(x) = u(x, 0)$ to find the constants A_0, A_n, and B_n. Plugging $t = 0$ into the general solution above, gives us

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

To calculate A_n, we integrate both sides. To calculate A_n, we multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate. To calculate B_n, we multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate. To simplify the integrals, we use the following, which are derived from trigonometric identities

$$\begin{aligned} \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx &= \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \neq 0 \\ 2L & \text{if } n = m = 0, \end{cases} \\ \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx &= \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \neq 0 \end{cases} \\ \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx &= 0. \end{aligned}$$

Our results are then

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \\ B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

1.4.5 Summary of Separation of Variables

Following is a summary the stuff involved in solving

$$X''(x) + \lambda X(x) = 0,$$

and the homogeneous heat equation. All this stuff should be memorized.

This gives the three Fourier series that need to be memorized. They correspond to the three types of boundary conditions noted in the first row. For PDE problems with different boundary conditions, we would get different Fourier series.

	Dirichlet BCs	Neumann BCs	Periodic BCs
Initial Conditions	$u(x,0) = f(x)$	$u(x,0) = f(x)$	$u(x,0) = f(x)$

Boundary Conditions	$X(0)=0,\ X(L)=0$	$X'(0)=0,\ X'(L)=0$	$X(-L)=X(L)$ $X'(-L)=X'(L)$
Eigenvalues, λ_n	$\left(\frac{n\pi}{L}\right)^2,\ n=1,2,3,\ldots$	$\left(\frac{n\pi}{L}\right)^2,\ n=0,1,2,3,\ldots$	$\left(\frac{n\pi}{L}\right)^2,\ n=0,1,1,2,2,3,3,\ldots$
Eigenfunctions, $X_n(x)$	$\sin\left(\frac{n\pi x}{L}\right),\ n=1,2,3,\ldots$	$\cos\left(\frac{n\pi x}{L}\right),\ n=0,1,2,3,\ldots$	$\cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right),\ X_0(x)=1$
Fourier Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$	$f(x)=A_0+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$	$f(x)=A_0+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ $+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx,$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$A_0=\frac{1}{2L}\int_{-L}^L f(x)\,dx,$ $A_n=\frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$ $B_n=\frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

1.4.6 Laplace's Equation on a Rectangle

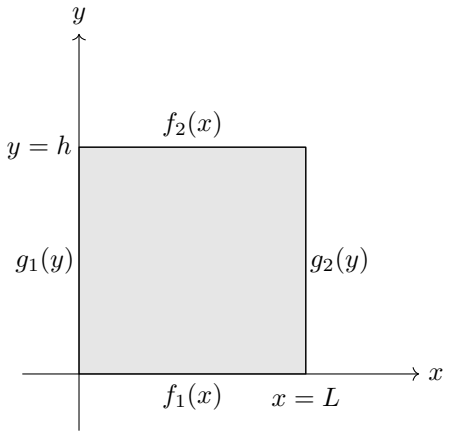
In this section, we solve a two-dimensional PDE. Recall that the Laplace equation is the equilibrium version of the heat equation. One way of interpreting it is as the equilibrium solution to a specific heat equation. In two dimensions, it is

$$\begin{aligned} \nabla^2 u &= 0 \\ u_{xx}+u_{yy} &= 0. \end{aligned}$$

We will first solve Laplace's equation on a rectangle, and later, solve it on a circular disk.

With Dirichlet boundary conditions, we are given the temperature on the four sides of the rectangle

$$\begin{aligned} u(x,0) &= f_1(x) \\ u(x,h) &= f_2(x) \\ u(0,y) &= g_1(y) \\ u(L,y) &= g_2(y). \end{aligned}$$



Separation of variables requires homogeneous boundary conditions, but ours are not. To overcome this, we'll treat three of the sides as $u=0$, and treat the fourth side as an initial condition. That is, we have to solve this PDE four times (once each for each side of the rectangle), then add them up to get the general solution.

In this section, we will only solve for one of the sides, leaving the other three parts of the solution as an exercise for the reader. We'll treat $u=0$ at the boundary except for the left side of the rectangle. That is, our PDE and the boundary conditions are

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \\ u(x,0) &= 0 \\ u(x,h) &= 0 \\ u(0,y) &= g_1(y) \\ u(L,y) &= 0. \end{aligned}$$

The boundary condition of $u(0,y)=g_1(y)$ will play the role of the initial condition $u(x,0)=f(x)$ of the previous heat equation problems.

Step 1

In step one, we separate the variables in the PDE. We want to write it in the separated form

$$u(x,y)=X(x)\cdot Y(y),$$

so we substitute this into the PDE, getting

$$X''(x)Y(y)+X(x)Y''(y)=0.$$

Rearranging gives

$$-\frac{X''}{X}=\frac{Y''}{Y}=-\lambda.$$

We set it equal to the constant $-\lambda$ since the two equations must be constant. This gives us the pair of ODEs

$$\begin{aligned} Y''+\lambda Y &= 0 \\ X''-\lambda X &= 0. \end{aligned}$$

Step 2

In step two, we solve the eigenvalue problem, that is, the homogeneous direction. In this case, the homogeneous

direction is the y direction since the top and bottom of the rectangle have $u = 0$. So we need to solve the ODE

$$\begin{aligned} Y'' + \lambda Y &= 0 \\ Y(0) &= 0 \\ Y(L) &= 0. \end{aligned}$$

We've already solved this differential equation when solving the homogeneous heat equation with Dirichlet boundary conditions. The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2, \qquad n = 1, 2, 3, \ldots.$$

The eigenfunctions are

$$Y_n(y) = \sin\left(\frac{n\pi y}{H}\right).$$

Step 3

In step three, we solve the remaining ODE.

$$X'' - \left(\frac{n\pi}{H}\right)^2 X = 0.$$

The general solution can be written in the following forms

$$\begin{aligned} X(x) &= C_1 e^{\frac{n\pi x}{H}} + C_2 e^{-\frac{n\pi x}{H}} \\ &= C_1 \cosh\left(\frac{n\pi x}{H}\right) + C_2 \sinh\left(\frac{n\pi x}{H}\right). \end{aligned}$$

By replacing x with $x - L$ we have another linearly independent set of solutions, which end up being more convenient

$$X(x) = C_1 \cosh\left(\frac{n\pi(x-L)}{H}\right) + C_2 \sinh\left(\frac{n\pi(x-L)}{H}\right).$$

From the boundary conditions, we know that $X(L) = 0$, which implies $C_1 = 0$, so the solution simplifies to

$$X_n(x) = \sinh\left(\frac{n\pi(x-L)}{H}\right).$$

Step 4

In step four, we write the solution to the PDE as the linear combinations of the products

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(x-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

This is the general solution to Laplace's equation on a rectangle with $u = 0$ along the top, bottom, and the right side.

Step 5

In the final step, we use the "initial condition" $u(0, y) = g_1(y)$ to find the coefficients C_n in the general solution. We start by setting $x = 0$

$$g_1(y) = u(0,y) = \sum_{n=1}^{\infty} C_n \sinh\left(-\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

Now, $C_n \sinh\left(-\frac{n\pi L}{H}\right)$ is just a number, and we treat the whole thing as the coefficient B_n in the Fourier sine series. So using the integral formula to compute B_n for the Fourier sine series, we get

$$C_n \sinh\left(\frac{n\pi L}{H}\right) = -\frac{2}{H}\int_0^H g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy.$$

Rearranging, gives us the coefficient C_n

$$C_n = -\frac{2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy.$$

In step four we found the general solution, and here we found the formula for C_n, so we now have the full solution. This is the solution to the Laplace equation of *one* side of the rectangle. For the full solution over the rectangle, we would have to find the solutions for the other three sides as well.

1.4.7 Laplace's Equation on a Disk

Here we solve Laplace's equation

$$\Delta u = 0,$$

over a disk of radius a. In other words,

$$\Delta u(x,y) = 0, \quad \text{if } x^2 + y^2 < a^2.$$

The temperature along the outer boundary of the disk, that is, the boundary condition, is given by a function

$$u(x,y) = f(x,y) \quad \text{if } x^2 + y^2 = a^2.$$

To separate the variables, we need to separate them in polar coordinates. First, we need to rewrite our equation and the boundary conditions in polar coordinates.

$$\begin{aligned} \Delta u(r,\theta) &= 0, \quad \text{if } r < a \\ u(a,\theta) &= f(\theta). \end{aligned}$$

Recall that the Laplacian of u in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{1.5}$$

where $0 \le r \le a$ and the most convenient choice for the bounds on θ is $-\pi \le \theta \le \pi$.

This is now sort of like a rectangle since r goes from 0 to a and θ goes from $-\pi$ to π. The region must be rectangle-like in order to separate variables.

Step 1

In step one, we separate variables. We need to write u in the form

$$u = R(r) \cdot \Theta(\theta),$$

where $R(r)$ is a function only of r and $\Theta(\theta)$ is a function only of θ. Substituting this into Eq. (1.5), we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Multiplying both sides by $\frac{r^2}{R\Theta}$ gives us

$$\begin{aligned} \frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} &= 0 \\ \frac{r^2R''}{R} + \frac{rR'}{R} &= -\frac{\Theta''}{\Theta} = \lambda. \end{aligned}$$

Since we have a function in one variable and its derivatives equal to a function in another variable and its derivatives, both functions must be constants, hence the constant λ on the right side.

So we get the pair of ODEs

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ r^2R'' + rR' - \lambda R &= 0. \end{aligned}$$

Note that θ is the same at $-\pi$ and π, so the θ part has periodic boundary conditions. Periodic BCs are homogeneous, so θ is the homogeneous direction. If the homogeneity was in the r direction, the resulting ODEs are difficult to solve and yield Bessel functions.

Step 2

The second step is to solve the eigenvalue problem, which is the ODE in the homogeneous direction. Since the homogeneity is in the θ direction, the eigenvalue problem is

$$\Theta'' + \lambda\Theta = 0,$$

and the boundary conditions are

$$\begin{aligned} \Theta(-\pi) &= \Theta(\pi) \\ \Theta'(-\pi) &= \Theta'(\pi). \end{aligned}$$

This is just the standard eigenvalue problem with periodic boundary conditions. It is like the one we solved with the heat equation, but in this case, $L = \pi$, so the eigenvalues are

$$\lambda_n = n^2, \quad n = 0, 1, 1, 2, 2, 3, 3, \ldots,$$

and the eigenfunctions are

$$\Theta_n(\theta) = \cos n\theta, \quad \sin n\theta,$$

with one particular eigenfunction being

$$\Theta_0(\theta) = 1.$$

Step 3

The third step is to solve the other equation, in this case,

$$r^2R'' + rR' - n^2R = 0.$$

Like with the heat equation, this is again a second order ODE so we expect two linearly independent solutions, but it is not a constant coefficient one. Notice that the powers on the r coefficients equals the number of derivatives on R. This is an example of a **Cauchy**, **Euler**, **Cauchy-Euler**, or **equidimensional** ODE. . For this type of ODE, we look for solutions of the form

$$R(r) = r^\alpha.$$

Plugging it into the ODE, we get

$$\begin{aligned} r^2(\alpha(\alpha-1)r^{\alpha-2}) + r(\alpha r^{\alpha-1}) - n^2(r^\alpha) &= 0 \\ (\alpha^2 - \alpha)r^\alpha + \alpha r^\alpha - n^2r^\alpha &= 0 \\ \alpha^2 r^\alpha - n^2 r^\alpha &= 0 \\ \alpha^2 - n^2 &= 0 \\ \alpha &= \pm n. \end{aligned}$$

So our two solutions are r^n and r^{-n}. When $n \neq 0$, our general solution is

$$R_n(r) = C_1 r^n + C_2 r^{-n}.$$

When $n = 0$, we have only one solution because $r^0 = r^{-0}$. We must find a second solution using another method, such as reduction of order. In this case, when $n = 0$, the general solution is

$$R_0(r) = C_1 + C_2 \ln r.$$

However, when $r = 0$, that is, we are at the center of the disk, we require that $R(0)$ be finite, that is, $|R(0)| < \infty$. This requirement ensures that the "boundary" $r = 0$ at the center of the disk is homogeneous. We have to do this because the outer boundary of the disk is not homogeneous, and we can only handle one nonhomogeneity at a time. In order for $R(0)$ to be bounded, the solution cannot have either r^{-n} or $\ln r$ at $r = 0$, which implies that $C_2 = 0$ in our general solution. So our general solution simplifies to

$$R_n(r) = C_1 r^n, \qquad R_0(r) = C_1.$$

Step 4

In step four, we combine the solutions to the separated ODEs to get the general solution to the original PDE. As usual, we write the $n = 0$ case separately as A_0.

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

Step 5

In the final step, we use the last (nonhomogeneous) condition. If time is involved, this will be the initial conditions. If only space is involved, this will just be the remaining boundary condition. In this case, the remaining condition is the boundary condition $u(a,\theta) = f(\theta)$, which gives the value of u along the outer boundary of the as a function of θ. In this case, we plug $r = a$ into the above equation to get

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta.$$

To find the coefficients, we use the same integral formulas as before, but we consider $A_n a^n$ and $B_n a^n$ as the coefficients instead of just A_n and B_n. Remember that $L = \pi$ in this case. Using the integral formulas, our coefficients are given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \\ A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta. \end{aligned}$$

Example Problem

Find the solution to

$$\begin{aligned} \Delta u(x,y) &= 0, \quad \text{when } x^2+y^2 \le 1 \\ u(x,y) &= y^3, \quad \text{when } x^2+y^2 = 1. \end{aligned}$$

This is the Laplace equation over the unit disk with $u = y^3$ on the outer boundary of the disk.

We know what the solution looks like, but we first have to convert the given boundary conditions to polar coordinates using $y = r\sin\theta$. The boundary conditions in polar coordinates are $f(\theta) = \sin^3\theta$ on the boundary $r = 1$.

We need the full fourier series for $\sin^3\theta$. We could compute it using the integral formulas given above, but it's much easier if we rewrite the sine function as exponentials.

$$\begin{aligned} \sin^3\theta &= (\sin\theta)^3 \\ &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 \\ &= \frac{e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}}{-8i} \\ &= \frac{-1}{4}\cdot\frac{e^{3i\theta}-e^{-3i\theta}}{2i} + \frac{3}{4}\cdot\frac{e^{i\theta}-e^{-i\theta}}{2i} \\ &= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta. \end{aligned}$$

This is a sine series for $\sin^3\theta$ so it must be *the* Fourier series for $\sin^3\theta$. So we have that

$$f(\theta) = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta.$$

We could also just have looked up a table of trigonometry identities.

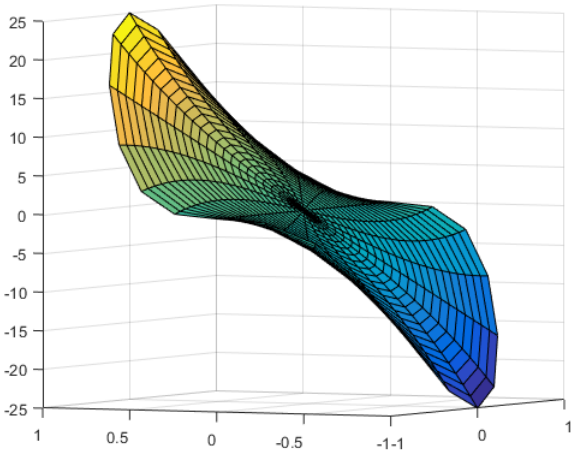
To get the general solution, we just have to plug the r^n into the Fourier series

$$u(r,\theta) = -\frac{1}{4}r^3\sin 3\theta + \frac{3}{4}r\sin\theta.$$

To convert this back into cartesian coordinates, we can use $r\sin\theta = y$ and $r = \sqrt{x^2+y^2}$ and the trig identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

$$\begin{aligned} u(r,\theta) &= -\frac{1}{4}r^3(3\sin\theta - 4\sin^3\theta) + \frac{3}{4}r\sin\theta \\ u(r,\theta) &= -\frac{3}{4}r^3\sin\theta + r^3\sin^3\theta + \frac{3}{4}r\sin\theta \\ u(x,y) &= -\frac{3}{4}(x^2+y^2)y + y^3 + \frac{3}{4}y \end{aligned}$$

A plot of the function is shown below:



Tip:
To check your final solution, just plug it back into the original PDE as well as any boundary conditions.

1.4.8 Harmonic Functions

A function u is a **harmonic function** if it satisfies $\Delta u = 0$.

An important property of harmonic functions is the **mean value theorem** which states that the value of a harmonic function at any point is always equal to the average value of the function on any circle centered at that point. A consequence of this is the **maximum principle** which states that harmonic functions have no local extrema (unless the function is a constant) within

the boundary. They may have extrema on the boundary. Consider the graph of the harmonic function of the previous section. Its maximum and minimum occur on the boundary of the circle $r = 1$.

Since a harmonic function u of x and y satisfies

$$u_{xx} + u_{yy} = 0,$$

it must be that at any point, the second x derivative is equal in magnitude and opposite in sign to the second y derivative. So essentially, every point in a harmonic function is a saddle-like point.

1.5 Fourier Series

1.5.1 Full Fourier Series

The full Fourier series of a function $f(x)$ defined on $[-L, L]$ is

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\ a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

In order for the series to be defined, the integrals for the coefficients a_n and b_n must be exist, and they will when $f(x)$ is integrable, that is, when

$$\int_{-L}^{L} |f(x)|\, dx < \infty.$$

Even when the series exists, it does not necessarily converge to $f(x)$. For a given function $f(x)$ whose series exists, the series may or may not converge for any particular value of x, and even if it does converge, it may or may not converge to $f(x)$. Usually, however, it converges.

What conditions will guarantee that the series behaves nicely? The standard condition is that the series behaves nicely if $f(x)$ is piecewise smooth. That is, we can divide the interval $[-L, L]$ into small intervals, such that on each subinterval, $f(x)$ is continuous and its derivative exists and is continuous. In other words, on $[-L, L]$, $f(x)$ cannot have any asymptotes, vertical lines (i.e. derivative doesn't exist), and they cannot be highly oscillatory (such as the graph of $\sin(2\pi \ln x)$.

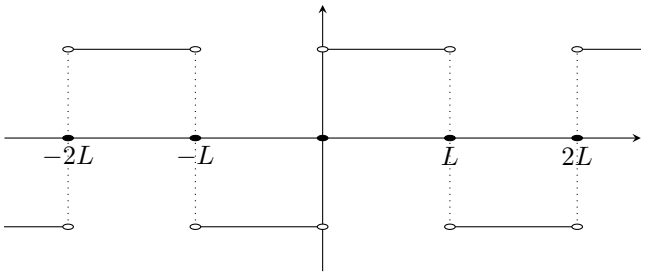
If $f(x)$ is piecewise smooth, then its Fourier series converges to the periodic extension of $f(x)$ at every point x at which $f(x)$ is continuous and $\frac{1}{2}(f(x^+) + f(x^-))$ at points x at which $f(x)$ is not continuous. In other words, if there is a discontinuity at $x = a$, then the value of the Fourier series at that point is halfway between the value that $f(x)$ approaches from the left at $x = a$ and the value as it approaches from the right.

To graph full Fourier series for a function $f(x)$ defined on $[-L, L]$, start by graphing $f(x)$ on $[-L, L]$. Then just do the periodic extension of that graph, that is, copy it to the left and right. At any discontinuities, plot the defined point, which will be half way between the limit of the function as it approaches the discontinuity from the left and from the right.

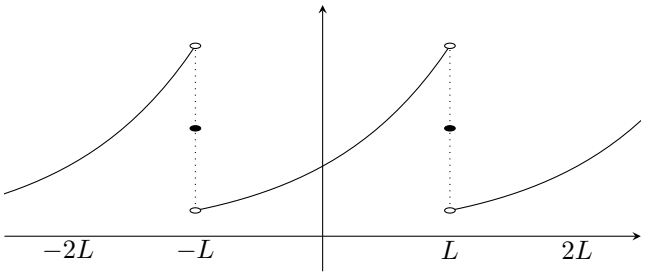
Below is a graph of the Fourier series of the piecewise function

$$f(x) = \begin{cases} 1 & 0 < x < L \\ -1 & -L < x < 0. \end{cases}$$

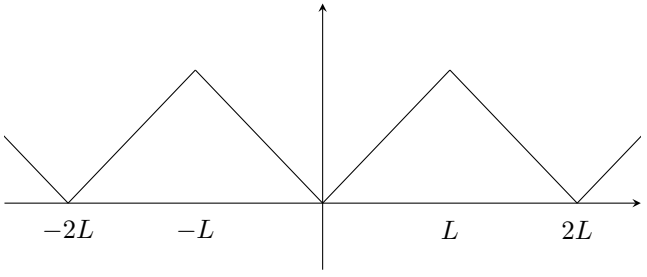
Notice that it has discontinuities at integer multiples of L. At those points, the value of the Fourier series is 0 since the value approaching those points from the left is 1 and the value $f(x)$ approaches those points from the right is -1.



Below is a graph of the Fourier series of the function $f(x) = e^x$ defined on $[-L, L]$. Notice that it has discontinuities at every other multiple of L. On the left side of a discontinuity, the function is approaching the value e^L. On the right side it is approaching the value e^{-L}. So at the discontinuities the Fourier series has the value $\frac{1}{2}(e^L + e^{-L}) = \cosh L$.



Below is a graph of the Fourier series of $f(x) = |x|$ defined on $[-L, L]$. Notice that it has no discontinuities.



We can also express the Fourier series of a function

$f(t)$ defined with a period τ as

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \\ a_0 &= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt \\ a_n &= \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\cos(n\omega t)\,dt \\ b_n &= \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\sin(n\omega t)\,dt. \end{aligned}$$

This is the same as the earlier expression except we've made the substitutions $x = t$, $\tau = 2L$ and $\tau = \frac{2\pi}{\omega}$ and we start the index at $n = 0$ instead of $n = 1$, which brings the a_0 term inside the summation. We can disregard the b_0 term since it is always zero.

At discontinuities of a Fourier series, something called **Gibbs phenomenon** occurs. This is where the graph of a finite number of terms of a Fourier series overshoots the actual graph of the Fourier series. This overshoot occurs at jump discontinuities and the amount of overshoot is about 9% of the height of the jump discontinuity. The more terms you add, the narrower the overshoot becomes, and the closer it moves to the discontiuity. However, the overshoot remains no matter how many terms you plot. In the limit, however, the Fourier series still converges for each point.

1.5.2 Even and Odd Functions

A function $f(x)$ is even if $f(-x) = f(x)$. Graphically, this means that if the function to the right of the origin is reflected about the vertical axis, then it will lie right on top of itself again. Examples of even functions include x^2 and $\cos x$.

A function $f(x)$ is odd if $f(-x) = -f(x)$. Graphically, this means that if the function to the right of the origin is rotated 180° counterclockwise, it will lie right on top of itself again. Examples of odd functions include x^3 and $\sin x$.

When it comes to functions:

$$\begin{aligned} \text{even} * \text{even} &= \text{even} \\ \text{even} * \text{odd} &= \text{odd} \\ \text{odd} * \text{odd} &= \text{even}. \end{aligned}$$

In a Fourier series,

$$\sin\left(\frac{n\pi x}{L}\right),$$

is odd and the other terms are all even. Odd functions have odd terms and even functions have even terms.

If $f(x)$ is odd then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

In particular, if $f(x)$ is an odd function then

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0 \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0. \end{aligned}$$

The second one is zero because $\cos x$ is an even function and an even function times an odd function $f(x)$ is an odd function. So the full Fourier series of an odd function has all of its a_n's equal to zero.

If $f(x)$ is even, then

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0,$$

since $\sin x$ is odd and an odd function times an even function is an odd function. So the full Fourier series of an odd function has all of its a_n's equal to zero.

If $f(x)$ is even, then the area under the curve on $[-L, L]$ is twice the area under the curve on $[0, L]$. That is,

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

So if $f(x)$ is odd, since $\sin x$ is odd, their product is even, and so

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Similarly, if $f(x)$ is even, since $\cos x$ is even, their product is even, and so

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

In summary, if $f(x)$ is odd, then the full Fourier series of $f(x)$ simplifies to the sine series. If $f(x)$ is even, then the full Fourier series simplifies to the cosine series.

1.5.3 Fourier Sine Series

The equation for the Fourier sine series of $f(x)$ defined on $[0, L]$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

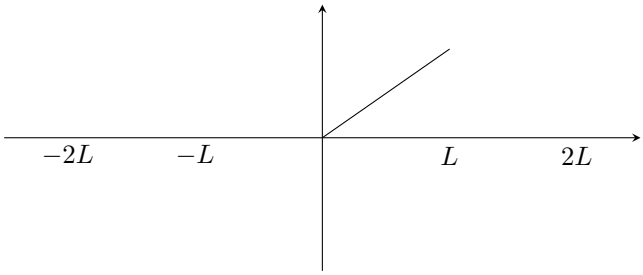
where the coefficients are

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

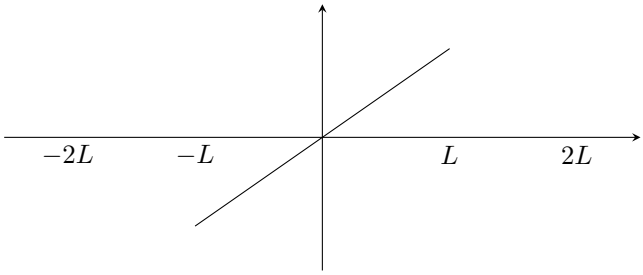
To graph a Fourier sine series of $f(x)$ on $[0, L]$,

1. Graph $f(x)$ on $[0, L]$
2. Graph the odd extension of the above on $[-L, 0]$
3. Graph the periodic extension of the above two to the left and right
4. Plot the points at the discontinuities

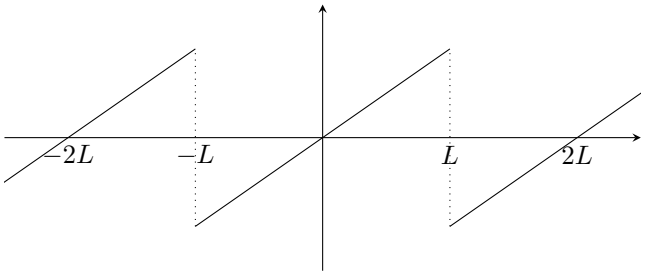
For example, to graph the Fourier sine series of $f(x) = x$ defined on $[0, L]$, we start by graphing x on $[0, L]$



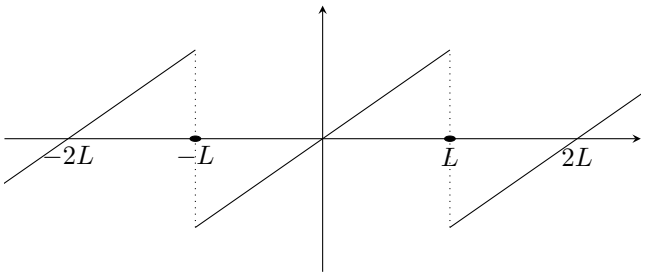
Next, we graph the odd extension of the above onto $[0, L]$, which is essentially taking the above graph and rotating it 180° about the origin.



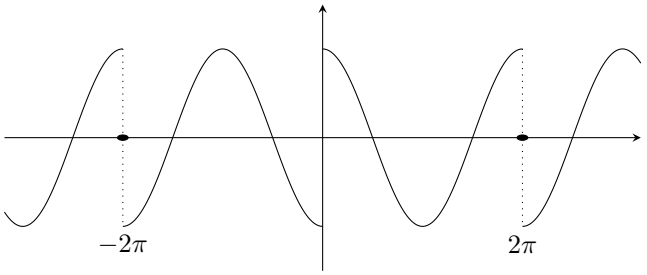
Next, we graph the periodic extension of what we graphed on $[-L, L]$, to the left and right.



Finally, we plot the points at the discontinuities.



The full Fourier series of an odd function $f(x)$ is the same as the sine series of $f(x)$. However, the sine series is its own thing. You can just as well plot the sine series for an even function, since you only plot $f(x)$ on $[0, L]$, and then you do the odd extension of that on $[-L, 0]$ before doing the periodic extension to the left and right. For example, the cosine series of $\cos x$ on $[0, 2\pi]$ is just the graph of $\cos x$. However, the sine series of $\cos x$ on $[0, 2\pi]$, looks like this:



Notice that on $[0, 2\pi]$ it is the graph of $\cos x$, but on $[-2\pi, 0]$, it is the odd extension of the graph that is on $[0, 2\pi]$. Then the picture on $[-2\pi, 2\pi]$ is repeated to the left and right.

To get the odd extenions of $f(x)$, we can compute $-f(-x)$.

Example:

Find the Fourier series for the function $f(t)$ with period τ defined in $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$

$$f(t) = \begin{cases} -1, & -\frac{\tau}{2} < t < 0 \\ +1, & 0 < t < \frac{\tau}{2} \end{cases}.$$

Since $f(t)$ is an odd function, we can disregard the a_n terms since they are all zero. Calculating the b_n terms gives us

$$\begin{aligned} b_n &= -\frac{2}{\tau}\int_{-\frac{\tau}{2}}^{0} \sin(n\omega t)\,dt + \frac{2}{\tau}\int_{0}^{\frac{\tau}{2}} \sin(n\omega t)\,dt \\ &= \frac{2}{\tau n\omega}\cos(n\omega t)\Big|_{-\frac{\tau}{2}}^{0} - \frac{2}{\tau n\omega}\cos(n\omega t)\Big|_{0}^{\frac{\tau}{2}} \\ &= \frac{4}{\tau n\omega} - \frac{4}{\tau n\omega}\cos(n\omega\frac{\tau}{2}) \\ &= \frac{2}{\pi n}[1-\cos(n\pi)] \\ &= \frac{2}{\pi n}[1-(-1)^n]. \end{aligned}$$

We can easily see that $b_0 = 0$, since its definition contains $\sin(n\omega t)$ and will therefore be zero for $n = 0$. So our Fourier series is

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin(n\omega t).$$

1.5.4 Fourier Cosine Series

The equation for the Fourier cosine series for a function $f(x)$ defined on $[0, L]$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

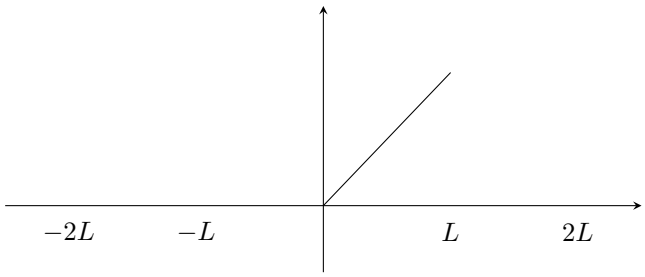
where the coefficients are

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

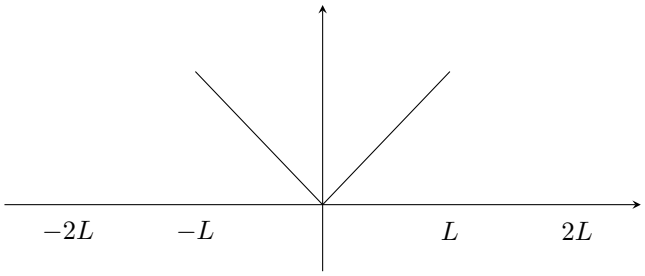
To graph a Fourier cosine series of $f(x)$ on $[0, L]$,

1. Graph $f(x)$ on $[0, L]$
2. Graph the even extension of the above on $[-L, 0]$
3. Graph the periodic extension of the above two to the left and right
4. Plot the points at the discontinuities

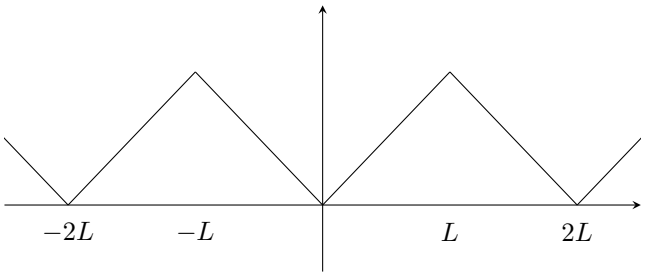
For example, to graph the Fourier cosine series of $f(x) = x$ defined on $[0, L]$, we start by graphing x on $[0, L]$



Next, we graph the even extension of the above onto $[0, L]$, which is essentially just reflecting the above graph about the vertical axis.



Next, we graph the periodic extension of what we graphed on $[-L, L]$, to the left and right.



Since there are no discontinuities, we are done.

The full Fourier series of an even function $f(x)$ is the same as the cosine series of $f(x)$. However, the cosine series is its own thing. You can just as well plot the cosine series for an odd function, since you only plot $f(x)$ on $[0, L]$, and then you do the even extension of that on $[-L, 0]$ before doing the periodic extension to the left and right.

Example:

Calculate the Fourier expansion for the triangular wave described by

$$f(x) = \begin{cases} \frac{b}{a}x + b & -a \le x < 0 \\ -\frac{b}{a}x + b & 0 \le x \le a \end{cases}.$$

Since this is an even function, we can use the shorter definition of the Fourier series since we know that all the b_n terms will be zero.

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \\ A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

In our case, $L = a$. Calculating the first term, we get

$$\begin{aligned} A_0 &= \frac{1}{a}\int_0^a \left(-\frac{b}{a}x + b\right)dx \\ &= -\frac{b}{a^2}\int_0^a x\,dx + \frac{b}{a}\int_0^a dx \\ &= -\frac{b}{2a^2}x^2\Big|_0^a + \frac{b}{a}x\Big|_0^a \\ &= \frac{b}{2}. \end{aligned}$$

Calculating the coefficients, we get

$$\begin{aligned} A_n &= \frac{2}{a}\int_0^a \left(-\frac{b}{a}x + b\right)\cos\left(\frac{n\pi x}{a}\right)dx \\ &= -\frac{2b}{a^2}\int_0^a x\cos\left(\frac{n\pi x}{a}\right)dx \\ &\qquad + \frac{2b}{a}\int_0^a \cos\left(\frac{n\pi x}{a}\right)dx \\ &= \frac{2b}{n^2\pi^2}\left[1 - \cos(n\pi)\right] \\ &= \frac{2b}{n^2\pi^2}\left[1 - (-1)^n\right]. \end{aligned}$$

We can evaluate the first integral by parts.

Our Fourier series is then

$$f(x) = \frac{b}{2} + \frac{2b}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos\left(\frac{n\pi x}{a}\right).$$

1.5.5 Even and Odd Parts of Functions

Given a function $f(x)$ defined on $[-L, L]$, the even part of the function is

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right],$$

and the odd part of the function is

$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right].$$

Notice that $f_e(x)$ as defined above is an even function and $f_o(x)$ is an odd function.

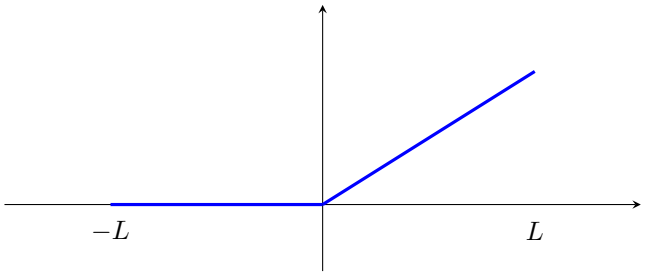
For example, $\cosh x$ is the even part of e^x, and $\sinh x$ is the odd part of e^x.

As another example, in the full Fourier series of a function $f(x)$, the terms with a_n coefficients form the even part of the Fourier series, and the sine terms form the odd part of the Fourier series. You can think of a full Fourier series of a function $f(x)$ as the sum of the cosine series of $f_e(x)$ and the sine series of $f_o(x)$. It is still the same amount of work. You still have to evaluate the same integrals.

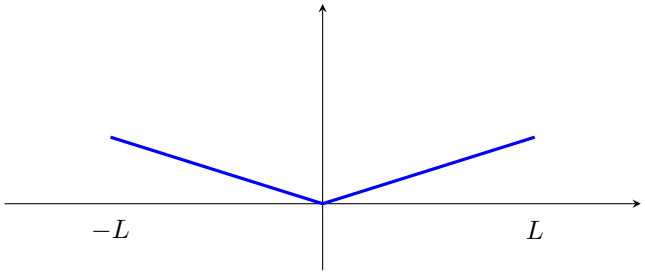
There is only one way to write a function as an even part and an odd part. Consider the function

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ x & 0 < x < L \end{cases},$$

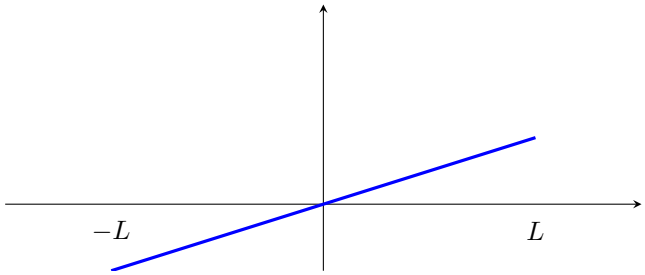
which has the following graph.



The even part of the function is $f_e(x) = \frac{1}{2}|x|$, shown below.



The odd part of the function is $f_o(x) = \frac{1}{2}x$ which is shown below.



When added together, the odd and even parts of the function become the original function.

1.5.6 Helpful Tips

It is essential to know the following identities, tips, and tricks when working with Fourier series and related functions.

These identities are necessary to solve some of the integrals that come up when solving for the Fourier coefficients:

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B\\
\sin(A-B) &= \sin A\cos B - \cos A\sin B\\
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
\cos(A-B) &= \cos A\cos B + \sin A\sin B\\
\cos^2 x &= \frac{1}{2}+\frac{1}{2}\cos 2x\\
\sin^2 x &= \frac{1}{2}-\frac{1}{2}\cos 2x\\
\sin 2x &= 2\sin x\cos x.
\end{aligned}$$

Notice that the last two can be derived from the first four.

For the following identities, n and m are nonnegative integers. Notice that in all the cases, the integral is zero if $n \neq m$. Knowing this, you can just plug in $n = m$ to find the nonzero possibilities for the integral as the integral will be greatly simplified when $n = m$, and you can use trig identities to solve the resulting integrals.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \neq m\\ \frac{L}{2} & \text{if } n = m \end{cases}$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \neq m\\ \frac{L}{2} & \text{if } n = m \neq 0\\ L & \text{if } n = m = 0, \end{cases}$$

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \neq m\\ L & \text{if } n = m \neq 0\\ 2L & \text{if } n = m = 0, \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \neq m\\ L & \text{if } n = m \neq 0 \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0.$$

The cosine Fourier series for a function $f(x)$ on $[0, L]$ has the form

$$f(x) = A_0 + A_1\cos\left(\frac{\pi x}{L}\right) + A_2\cos\left(\frac{2\pi x}{L}\right) + \cdots$$

The cosine series for $f(x) = 1$ is simply $f(x) = 1+0+0+0+\cdots$. That is, the cosine series of a constant function is just the constant. We need to be able to recognize that, so we don't waste undue time computing a Fourier series. The sine series for $\sin^2\left(\frac{\pi x}{L}\right)$ involves a messy integral, but the cosine series is just the trigonometric identity

$$\sin^2\left(\frac{\pi x}{L}\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right).$$

Notice that the trigonometry identity gives us a cosine series for $\sin^2\left(\frac{\pi x}{L}\right)$, therefore, it must be *the* cosine series for $\sin^2\left(\frac{\pi x}{L}\right)$.

The sine Fourier series for a function $f(x)$ on $[0, L]$ has the form

$$f(x) = B_1\sin\left(\frac{\pi x}{L}\right) + B_2\sin\left(\frac{2\pi x}{L}\right) + \cdots$$

Similarly, the sine series for $f(x) = \sin\left(\frac{5\pi x}{L}\right)$ is simply $f(x) = 0+0+0+0+\sin\left(\frac{5\pi x}{L}\right)+\cdots$. Just like the Taylor series of a polynomial is just the polynomial, the sine series of a sine series is just the function itself. In other words, if it is already a sine series then it is *the* sine series.

What are the values of the C_n in the following series?

$$\cos(2\pi y) = \sum_{n=0}^{\infty} C_n\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi y}{2}\right).$$

The solution is actually very simple. The key is in recognizing it as a Fourier cosine series. The right side is the typical Fourier cosine series but with $A_n = C_n\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)$. We could find A_n using the integral formula for Fourier cosine series, but there's no need to do that if we recognize the left as already a cosine series. In other words, we know that the right side must simplify to $\cdots 0+0+\cos(2\pi y)+0+0+\cdots$. Equating the two cosine expressions,

$$\cos(2\pi y) = A_n\cos\left(\frac{n\pi y}{2}\right),$$

we find that $n = 4$. In other words, $A_n = 1$ when $n = 4$ and $A_n = 0$ when $n \neq 4$. So

$$C_n = \begin{cases} 0, & \text{if } n \neq 4\\ \frac{1}{2\pi\sinh(2\pi)}, & \text{if } n = 4. \end{cases}$$

So the series simplifies to

$$\cos(2\pi y) = \cos(2\pi y).$$

Simplifications to keep in mind include

$$\begin{aligned}
\sin n\pi &= 0\\
\cos n\pi &= (-1)^n\\
e^{in\pi} &= (-1)^n\\
e^{-in\pi} &= (-1)^n.
\end{aligned}$$

1.5.7 Differentiation and Integration

Consider the function $f(x) = \frac{1}{2}x$ on the interval $[-\pi, \pi]$. Its Fourier series is

$$\frac{1}{2}x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

If we differentiate both sides, we get

$$\frac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \cos(nx).$$

However, this is not the cosine series for $\frac{1}{2}$. The actual cosine series for that constant is simply $\frac{1}{2} = \frac{1}{2} + 0 + 0 + \cdots$. Clearly, differentiating a Fourier series term-by-term is not always valid.

When is it okay to differentiate a Fourier series. In short, it is okay to differentiate a Fourier series if the graph of the series does not have any discontinuities. If the graph has discontinuities, then it is not differentiable.

In order for the Fourier series of a function $f(x)$ defined on $[-L, L]$ to be continuous, $f(x)$ must be continuous on $[-L, L]$, and $f(-L) = f(L)$.

In order for the Fourier cosine series of a function $f(x)$ defined on $[0, L]$ to be continuous, $f(x)$ must only be continuous on $[0, L]$. The cosine extension creates an even function, so it naturally takes care of $f(-L) = f(L)$.

In order for the Fourier sine series of a function $f(x)$ defined on $[0, L]$ to be continuous, $f(x)$ must be continuous on $[0, L]$, and $f(0) = f(L) = 0$.

The easiest way to be sure that a Fourier series has no discontinuities is to graph the function.

What about integrating? Integrating is never a problem, but the result is not necessarily another Fourier series. In general, integrating is 'nicer' than differentiating, since integrating a function makes it behave smoother and differentiating makes a function behave less smoothly in some sense. You can always integrate a function, but you cannot always differentiate a function.

In practice, when integrating a Fourier series term-by-term, it works better to do a definite integral than an indefinite integral.

Integrating

$$\frac{1}{2}x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx),$$

from $x = 0$ to $x = y$, we get

$$\begin{aligned}
\frac{1}{2}\int_0^y x\,dx &= \sum_{n=1}^{\infty} \int_0^y \frac{(-1)^{n+1}}{n} \sin(nx)\,dx \\
\frac{1}{4}y^2 &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)\Big|_0^y \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (\cos(ny) - 1) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(ny) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.
\end{aligned}$$

The first sum is the cosine series of $\frac{1}{4}y^2$ and the second sum is just a number—the constant associated with all cosine series. We know the integral formula for the constant of a cosine series, so we just use that to find its value

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{4}y^2\,dy = \frac{\pi^2}{12}.$$

So our final result in the form of a Fourier series is

$$\frac{1}{4}y^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(ny).$$

Integrating this result from $y = 0$ to $y = z$, we get

$$\begin{aligned}
\int_0^z \frac{1}{4}y^2\,dy &= \int_0^z \frac{\pi^2}{12}\,dy + \sum_{n=1}^{\infty} \int_0^z \frac{(-1)^n}{n^2} \cos(ny)\,dy \\
\frac{1}{12}z^3 &= \frac{\pi^2}{12}z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nz).
\end{aligned}$$

This is a true statement, but it is not a Fourier series, because the first term on the right contains a z. However, it is still a useful result because we can use it to create Fourier series. For example, subtracting that term from both sides gives us a Fourier series

$$\frac{1}{12}z^3 - \frac{\pi^2}{12}z = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nz).$$

Another possibility is to find the Fourier series of $\frac{\pi^2}{12}z$ and merge it with the other sum. Doing so, we get

$$\begin{aligned}
\frac{1}{12}z^3 &= \frac{\pi^2}{6}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nz)\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nz) \\
&= \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\frac{\pi^2}{6} + \frac{(-1)^n}{n^3}\right) \sin(nz) \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\pi^2}{6n} - \frac{1}{n^3}\right) \sin(nz).
\end{aligned}$$

1.5.8 Complex Fourier Series

The complex Fourier expansion for a function $f(x)$ defined on $[-L, L]$ is given by

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}} \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}}\, dx. \end{aligned}$$

This is equivalent to the full Fourier series of a function. Notice that the signs in the exponent are not the same in the sum and the integral.

Note: Some authors switch where the negative exponent appears.

The complex Fourier series is derived from the other Fourier series using the Euler relations

$$\begin{aligned} \cos\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) \\ \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right). \end{aligned}$$

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \\ &= a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) + b_n \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)\right] \\ &= a_0 + \sum_{n=1}^{\infty}\frac{1}{2}\left(a_n + ib_n\right)e^{-\frac{in\pi x}{L}} + \sum_{n=1}^{\infty}\frac{1}{2}\left(a_n - ib_n\right)e^{\frac{in\pi x}{L}} \\ &= a_0 + \sum_{n=1}^{\infty}\frac{1}{2}\left(a_n + ib_n\right)e^{-\frac{in\pi x}{L}} + \sum_{n=-\infty}^{-1}\frac{1}{2}\left(a_{-n} - ib_{-n}\right)e^{-\frac{in\pi x}{L}} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}}, \end{aligned}$$

$$c_n = \begin{cases} c_0 = a_0 & n = 0 \\ c_n = \frac{1}{2}\left(a_n + ib_n\right) & n = 1, 2, 3, \ldots \\ c_{-n} = \frac{1}{2}\left(a_{-n} - ib_{-n}\right) & n = -1, -2, -3, \ldots \end{cases}$$

Notice that a_{-n} and b_{-n} are only defined when n is negative.

Looking at the coefficients, we get

$$\begin{aligned} c_0 = a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{i(0)\pi x}{L}}\, dx \\ c_n = \frac{1}{2}\left(a_n + ib_n\right) &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + i\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}}\, dx \\ c_{-n} = \frac{1}{2}\left(a_{-n} - ib_{-n}\right) &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{-n\pi x}{L}\right)dx - i\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{-n\pi x}{L}\right)dx\right) \\ &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + i\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\ &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}}\, dx. \end{aligned}$$

Example:

Find the Fourier series for $f(x) = e^x$ on $[-L, L]$ using complex exponentials.

The formula for the complex form of the full Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}}.$$

The coefficients are

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} e^x\, e^{\frac{in\pi x}{L}}\, dx \\ &= \frac{1}{2L}\int_{-L}^{L} e^{(1+\frac{in\pi}{L})x}\, dx \\ &= \frac{1}{2L}\frac{e^{(1+\frac{in\pi}{L})x}}{1+\frac{in\pi}{L}}\Big|_{-L}^{L} \\ &= \frac{1}{2L+2in\pi}\left(e^{(L+in\pi)} - e^{-(L+in\pi)}\right). \end{aligned}$$

Example:

Find the Fourier series for $f(x) = x$ on $[-1, 1]$ using complex exponentials.

Plugging $L = 1$ into the formula for complex Fourier series, we get

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x} \\ c_n &= \frac{1}{2}\int_{-1}^{1} x\, e^{in\pi x}\, dx. \end{aligned}$$

To calculate the coefficients c_n, we use integration by parts

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} x\, e^{in\pi x}\, dx \\ &= \frac{x\, e^{in\pi x}}{2in\pi}\Big|_{-1}^{1} + \frac{e^{in\pi x}}{2n^2\pi^2}\Big|_{-1}^{1} \\ &= \frac{1}{in\pi}\left[\frac{e^{in\pi}+e^{-in\pi}}{2}\right] + \frac{i}{n^2\pi^2}\left[\frac{e^{in\pi}-e^{-in\pi}}{2i}\right] \\ &= \frac{1}{in\pi}\left[\frac{e^{in\pi}+e^{-in\pi}}{2}\right] \\ &= -\frac{i}{n\pi}\cos(n\pi) \\ &= \frac{i}{n\pi}(-1)^{n+1}. \end{aligned}$$

Notice that the $\frac{1}{2i}\left(e^{in\pi} - e^{-in\pi}\right)$ terms dropped out because it is the sine function, which is zero for all values of n.

Notice that the formula for c_n will not work when $n = 0$, so we have to calculate that term separately.

$$\begin{aligned} c_0 &= \frac{1}{2}\int_{-1}^{1} x\, dx \\ &= \frac{1}{4}x^2\Big|_{-1}^{1} \\ &= 0. \end{aligned}$$

So our Fourier series is

$$f(x) = \sum_{n=-\infty}^{-1} \frac{i(-1)^{n+1}}{n\pi} e^{-in\pi x} + \sum_{n=1}^{\infty} \frac{i(-1)^{n+1}}{n\pi} e^{-in\pi x}.$$

Rewriting the first summation with a positive index and noting that $(-1)^{-n+1} = (-1)^{n+1}$, we get

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{i(-1)^{n+1}}{-n\pi} e^{in\pi x} + \sum_{n=1}^{\infty} \frac{i(-1)^{n+1}}{n\pi} e^{-in\pi x} \\ f(x) &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\frac{1}{2i}(e^{in\pi x} - e^{-in\pi x}) \\ f(x) &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x). \end{aligned}$$

1.6 Eigenfunction Expansion

Here, we detail a very important method that everyone uses to solve PDEs. In principle, this method works for anything, unlike separation of variables which only works for homogeneous equations. This method can be used to solve nonhomogeneous PDEs, and with care, even nonlinear ones. This method works for the homogeneous case as well, but since it requires foreknowledge of which Fourier series to use, we teach separation of variables when first solving those.

1.6.1 Heat Equation with Dirichlet BCs

In this section we use the method to solve the nonhomogeneous heat equation

$$u_t - ku_{xx} = f(x,t),$$

on $[0, L]$ with Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= \alpha(t) \\ u(L,t) &= \beta(t) \\ u(x,0) &= g(x). \end{aligned}$$

Notice that the boundary conditions are no longer homogeneous, that is, equal to zero. Notice that u, u_t, u_{xx}, $f(x)$, and $g(x)$ are all functions of x (some also of t).

Step 1

In step one, we write all the functions of x as Fourier series. We use the Fourier series that we would get if we were doing the homogeneous problem. If this problem was homogeneous, then f, α, and β would all be zero. In that case, we would have Dirichlet boundary conditions, so we use the sine series.

$$u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients $\hat{u}_n(t)$ are now functions of t. For each time t, there is a different Fourier series, so our coefficients are functions of time. From the known formula for computing Fourier coefficients, we have that

$$\hat{u}_n(t) = \frac{2}{L}\int_0^L u(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx.$$

Our second Fourier series is

$$u_t(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

and the coefficients are

$$\begin{aligned} \hat{v}_n(t) &= \frac{2}{L}\int_0^L u_t(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L}\int_0^L \frac{\partial u}{\partial t}(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{d}{dt}\left(\frac{2}{L}\int_0^L u(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx\right) \\ &= \frac{d}{dt}\hat{u}_n(t). \end{aligned}$$

Notice that we were able to pull the derivative with respect to t outside the integral since the integral is with respect to x. In the end, these coefficients reduced to the time derivative of the coefficients for the Fourier series of $u(x,t)$.

Our third Fourier series is

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \hat{w}_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

with coefficients

$$\hat{w}_n(t) = \frac{2}{L}\int_0^L u_{xx}(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx.$$

This time, we cannot pull the derivative outside the integral since both the derivative and the integral are with respect to x. But we can get rid of the second partial derivative by integrating twice by parts. Integrating once by parts gives us

$$\begin{aligned} \hat{w}_n(t) &= \frac{2}{L} u_x(x,t) \sin\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{L} \\ &\quad -\frac{2}{L}\left(\frac{n\pi}{L}\right)\int_0^L u_x(x,t)\, \cos\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

Evaluating the first term gives us zero. Integrating the remaining integral by parts a second time gives us

$$\begin{aligned} \hat{w}_n(t) &= -\frac{2n\pi}{L^2} u(x,t) \cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{L} \\ &\quad -\frac{2}{L}\left(\frac{n\pi}{L}\right)^2\int_0^L u(x,t)\, \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

Here, we can evaluate the first term using the boundary conditions, and the second term is a coefficient we found earlier times a factor.

$$\hat{w}_n(t) = -\frac{2n\pi}{L^2}\left[(-1)^n\beta(t) - \alpha(t)\right] - \left(\frac{n\pi}{L}\right)^2 \hat{u}_n(t).$$

The Fourier series for the remaining two functions

and their coefficients are

$$\begin{aligned} f(x,t) &= \sum_{n=1}^{\infty} \hat{f}_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ \hat{f}_n(t) &= \frac{2}{L}\int_0^L f(x,t)\,\sin\left(\frac{n\pi x}{L}\right) dx \\ g(x) &= \sum_{n=1}^{\infty} \hat{g}_n \sin\left(\frac{n\pi x}{L}\right) \\ \hat{g}_n &= \frac{2}{L}\int_0^L g(x)\,\sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

Notice that since g is a function of only x, its coefficients are constants rather than functions of time.

Step 2

In step 2, we substitute all the Fourier series into the PDE, and rearrange them so they are all on the same side. Finally, since they're all sine series, we can factor the sine function out.

$$\begin{aligned} u_t - ku_{xx} - f(x,t) &= 0 \\ \sum_{n=1}^{\infty} \frac{d\hat{u}_n}{dt}\sin\left(\frac{n\pi x}{L}\right) - k\sum_{n=1}^{\infty}\left[-\frac{2}{L}\frac{n\pi}{L}\left[(-1)^n\beta(t)-\alpha(t)\right] - \left(\frac{n\pi}{L}\right)^2\hat{u}_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty}\hat{f}_n(t)\sin\left(\frac{n\pi x}{L}\right) &= 0 \\ \sum_{n=1}^{\infty}\left[\frac{d\hat{u}_n}{dt} - k\left[-\frac{2}{L}\frac{n\pi}{L}\left[(-1)^n\beta(t)-\alpha(t)\right] - \left(\frac{n\pi}{L}\right)^2\hat{u}_n(t)\right] - \hat{f}_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) &= 0 \\ \sum_{n=1}^{\infty}\left[\frac{d\hat{u}_n}{dt} + k\left(\frac{n\pi}{L}\right)^2\hat{u}_n(t) + \frac{2kn\pi}{L^2}\left[(-1)^n\beta(t)-\alpha(t)\right] - \hat{f}_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) &= 0 \end{aligned}$$

This must be equal to the sine series for 0. This implies that the inside must be equal to zero. So we can write

$$\frac{d\hat{u}_n}{dt} + k\left(\frac{n\pi}{L}\right)^2\hat{u}_n(t) = \hat{r}_n(t),$$

where

$$\hat{r}_n(t) = -\frac{2kn\pi}{L^2}\left[(-1)^n\beta(t)-\alpha(t)\right] + \hat{f}_n(t),$$

is a known function of t.

Step 3

In the previous two sections, we reduced the heat equation to the first-order linear constant-coefficient nonhomogeneous ODE

$$\dot{\hat{u}}_n + k\left(\frac{n\pi}{L}\right)^2\hat{u}_n(t) = \hat{r}_n(t),$$

where

$$\hat{r}_n(t) = -\frac{2kn\pi}{L^2}\left[(-1)^n\beta(t)-\alpha(t)\right] + \hat{f}_n(t),$$

is a known function of t, and the initial condition is

$$\hat{u}_n(0) = \hat{g}_n.$$

We will get a different function $u(x,t)$ for each n.

To solve this, we use the method of an **integrating factor** using the coefficient on $\hat{u}_n(t)$

$$\begin{aligned} \text{I.F.} &= e^{\int k\left(\frac{n\pi}{L}\right)^2 dt} \\ &= e^{k\left(\frac{n\pi}{L}\right)^2 t}. \end{aligned}$$

We don't have to worry about the constant of integration.

Next, we multiply the ODE by the integrating factor

$$e^{k\left(\frac{n\pi}{L}\right)^2 t}\dot{\hat{u}}_n + k\left(\frac{n\pi}{L}\right)^2 e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{u}_n(t) = e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{r}_n(t).$$

The left side is now the result of a product rule,

$$\frac{d}{dt}\left(e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{u}_n(t)\right) = e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{r}_n(t).$$

Now we just have to integrate. We can integrate by definite or indefinite integral, but it's easiest to use a definite integral from $t = 0$ to $t = t'$.

$$\begin{aligned} \int_0^{t'} \frac{d}{dt}\left(e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{u}_n(t)\right) dt &= \int_0^{t'} e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{r}_n(t)\,dt \\ e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{u}_n(t)\Big|_0^{t'} &= \int_0^{t'} e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{r}_n(t)\,dt \\ e^{k\left(\frac{n\pi}{L}\right)^2 t'}\hat{u}_n(t') - \hat{u}_n(0) &= \int_0^{t'} e^{k\left(\frac{n\pi}{L}\right)^2 t}\hat{r}_n(t)\,dt. \end{aligned}$$

This is as far as we can go without knowing $\alpha(t)$, $\beta(t)$, and $f(x,t)$.

Example

Solve the PDE

$$u_t = ku_{xx} + x,$$

defined on $0 \le x \le 1$ using eigenfunction expansion. The boundary and initial conditions are

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 1 \\ u(x,0) &= g(x). \end{aligned}$$

We use the sine series because we have Dirichlet-like boundary conditions. Since we already solved most of it in the previous section, we basically just have to plug stuff in.

First, we have to find the fourier series for $f(x,t) = x$, which is

$$f(x,t) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x),$$

in this case. For this specific problem, we have that

$$\begin{aligned} L &= 1 \\ \beta(t) &= 1 \\ \alpha(t) &= 0 \\ \hat{f}_n(t) &= \frac{2(-1)^{n+1}}{n\pi} \\ \hat{u}_n(0) &= \hat{g}_n. \end{aligned}$$

Plugging these into the results we got in the previous section, we get

$$\begin{aligned} e^{k(n\pi)^2 t'}\hat{u}_n(t') - \hat{g}_n &= \int_0^{t'} e^{k(n\pi)^2 t}\hat{r}_n(t)\,dt \\ e^{k(n\pi)^2 t'}\hat{u}_n(t') - \hat{g}_n &= 2(-1)^{n+1}\left(kn\pi + \frac{1}{n\pi}\right)\int_0^{t'} e^{k(n\pi)^2 t}\,dt \\ e^{k(n\pi)^2 t'}\hat{u}_n(t') - \hat{g}_n &= 2(-1)^{n+1}\left(kn\pi + \frac{1}{n\pi}\right)\frac{1}{k(n\pi)^2} e^{k(n\pi)^2 t}\Big|_0^{t'} \\ e^{k(n\pi)^2 t'}\hat{u}_n(t') - \hat{g}_n &= 2(-1)^{n+1}\left(kn\pi + \frac{1}{n\pi}\right)\frac{1}{k(n\pi)^2}\left(e^{k(n\pi)^2 t'} - 1\right) \\ \hat{u}_n(t') &= 2(-1)^{n+1}\left(kn\pi + \frac{1}{n\pi}\right)\frac{1}{k(n\pi)^2}\left(1 - e^{-k(n\pi)^2 t'}\right) + \hat{g}_n e^{-k(n\pi)^2 t'} \\ \hat{u}_n(t') &= 2(-1)^{n+1}\left(\frac{1}{n\pi} + \frac{1}{kn^3\pi^3}\right) + \left(\hat{g}_n - \frac{2(-1)^{n+1}}{n\pi} - \frac{2(-1)^{n+1}}{kn^3\pi^3}\right) e^{-k(n\pi)^2 t'} \end{aligned}$$

So our solution is

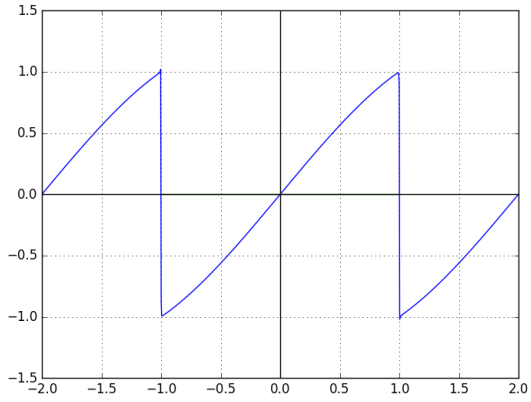
$$u(x,t) = \sum_{n=1}^{\infty} 2(-1)^{n+1}\left(\frac{1}{n\pi} + \frac{1}{kn^3\pi^3}\right)\sin(n\pi x) + \sum_{n=1}^{\infty}\left(\hat{g}_n - \frac{2(-1)^{n+1}}{n\pi} - \frac{2(-1)^{n+1}}{kn^3\pi^3}\right) e^{-k(n\pi)^2 t'}\sin(n\pi x).$$

Notice that we have separated the solution into two series—one that is a function of time and one that is not. The series containing t will decay to zero as $t \to \infty$ since t is a negative power in the exponential. This is called the **transient** part of the solution, and is the only part in which the initial conditions play a part. After a long time, only the first series, the equilibrium part of the solution will remain.

To check that our solution is reasonable, we cannot plugin in the boundary conditions and expect them to be true. This is because a Fourier series may approach the boundary values near the boundaries, but if there's a discontinuity right at the boundary, the Fourier series will give a different answer. One possibility is to graph the first 50 or so terms of the Fourier series solution to see if it satisfies the boundary values very near the boundaries.

In general, we also cannot differentiate the Fourier series solution with respect to t and twice with respect to x since there will often be discontinuities, which make differentiation invalid. Even if we end up with a cosine series solution that appears to be continuous and therefore differentiable, we won't be able to differentiate it twice, because the first derivative will be a sine series, which will most likely have discontinuities.

Graphing only the first series—the equilibrium solution—gives us the following graph. From this graph, we can at least verify that the boundary conditions appear to be verified.



We can calculate the equilibrium solution directly from the PDE. Since $u_t = 0$ at equilibrium, the PDE simplifies to the separable ODE

$$\frac{d^2u}{dx^2} = -\frac{1}{k}x,$$

which has the solution

$$u_{eq}(x) = -\frac{1}{6k}x^3 + C_1 x + C_2.$$

The boundary conditions tell us that

$$\begin{aligned} C_1 &= 1 + \frac{1}{6k} \\ C_2 &= 0, \end{aligned}$$

So the equilibrium solution is

$$u_{eq}(x) = -\frac{1}{6k}x^3 + \left(1 + \frac{1}{6k}\right)x.$$

Comparing this result with the graph above, we see that it is a perfect match on $[0, 1]$, so at least we know that the equilibrium part of the solution is correct.

1.6.2 Heat Equation with Neumann BCs

In this section we use the eigenfunction expansion method to solve the nonhomogeneous heat equation

$$u_t = ku_{xx} + f(x,t),$$

on $[0, L]$ with Neumann-like boundary conditions

$$\begin{aligned} u_x(0,t) &= \alpha(t) \\ u_x(L,t) &= \beta(t) \\ u(x,0) &= g(x). \end{aligned}$$

Since we have Neumann-like boundary conditions, we will be using cosine series.

Step 1

Computing our Fourier series, we get

$$\begin{aligned} u(x,t) &= \hat{u}_0(t) + \sum_{n=1}^{\infty} \hat{u}_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ \hat{u}_0(t) &= \frac{1}{L}\int_0^L u(x,t)\,dx \\ \hat{u}_n(t) &= \frac{2}{L}\int_0^L u(x,t)\,\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Since the cosine series above is a continuous function, we can simply differentiate it with respect to t to get the series for u_t.

$$u_t(x,t) = \dot{\hat{u}}_0(t) + \sum_{n=1}^{\infty} \dot{\hat{u}}_n(t) \cos\left(\frac{n\pi x}{L}\right).$$

To get the series for u_{xx}, we can't just differentiate the series for $u(x,t)$ twice with respect to x, because we don't know that the boundary values equal each other for u_x.

$$\begin{aligned} u_{xx}(x,t) &= \hat{w}_0(t) + \sum_{n=1}^{\infty} \hat{w}_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ \hat{w}_0(t) &= \frac{1}{L}\int_0^L u_{xx}(x,t)\,dx \\ &= \frac{1}{L}\left(\beta(t) - \alpha(t)\right) \\ \hat{w}_n(t) &= \frac{2}{L}\int_0^L u_{xx}(x,t)\,\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left[(-1)^n\beta(t) - \alpha(t)\right] - \frac{n^2\pi^2}{L^2}\hat{u}_n(t). \end{aligned}$$

The Fourier series for the remaining two functions and their coefficients are

$$\begin{aligned} f(x,t) &= \hat{f}_0(t) + \sum_{n=1}^{\infty} \hat{f}_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ \hat{f}_0(t) &= \frac{1}{L}\int_0^L f(x,t)\,dx \\ \hat{f}_n(t) &= \frac{2}{L}\int_0^L f(x,t)\,\cos\left(\frac{n\pi x}{L}\right)dx \\ g(x) &= \hat{g}_0 + \sum_{n=1}^{\infty} \hat{g}_n \cos\left(\frac{n\pi x}{L}\right) \\ \hat{g}_0 &= \frac{1}{L}\int_0^L g(x)\,dx \\ \hat{g}_n &= \frac{2}{L}\int_0^L g(x)\,\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Step 2

In the next step, we substitute these all into the original PDE

$$\begin{aligned} u_t - ku_{xx} - f(x,t) &= 0 \\ \dot{\hat{u}}_0(t) - \frac{k}{L}\left(\beta(t) - \alpha(t)\right) - \hat{f}_0(t) + \sum_{n=1}^{\infty}\left[\dot{\hat{u}}_n(t) + \frac{kn^2\pi^2}{L^2}\hat{u}_n(t) - \frac{2k}{L}\left[(-1)^n\beta(t) - \alpha(t)\right] - \hat{f}_n(t)\right]\cos\left(\frac{n\pi x}{L}\right) &= 0 \end{aligned}$$

Since this is the cosine series for 0, the constants on the left must be the constant term of the cosine series for 0 and the contents of the brackets must be the coefficients of the cosine series of zero. Both of these must be equal

to zero. That is,

$$\begin{aligned}\dot{\hat{u}}_n(t) + \frac{kn^2\pi^2}{L^2}\hat{u}_n(t) - \frac{2k}{L}\left[(-1)^n\beta(t) - \alpha(t)\right] - \hat{f}_n(t) &= 0\\ \dot{\hat{u}}_0(t) - \frac{k}{L}\left(\beta(t) - \alpha(t)\right) - \hat{f}_0(t) &= 0.\end{aligned}$$

Example

Now, consider the specific conditions

$$\begin{aligned}\alpha(t) &= 0\\ \beta(t) &= 0\\ L &= 1\\ k &= 1\\ f(x,t) &= 1\\ g(x) &= 0.\end{aligned}$$

The cosine series of 1 is just $\hat{f}_0(t) = 1$. All the other coefficients $\hat{f}_n(t)$ are zero. Plugging these into the equations gives us

$$\begin{aligned}\dot{\hat{u}}_n(t) + n^2\pi^2\hat{u}_n(t) &= 2(-1)^{n+1}\\ \dot{\hat{u}}_0(t) &= 0.\end{aligned}$$

We know that the general solution is

$$u(x,t) = \hat{u}_0(t) + \sum_{n=1}^{\infty}\hat{u}_n(t)\cos(n\pi x),$$

and now we know that $\dot{\hat{u}}_0(t) = 0$ so $\hat{u}_0(t) = C$, but from the initial conditions, we get $C = 0$. We also have a first order differential equation in terms of $\hat{u}_n(t)$. Our integrating factor is

$$\text{I.F.} = e^{\int n^2\pi^2\,dt} = e^{n^2\pi^2 t}.$$

Multiplying the ODE by the integrating factor and solving, we get

---

$$\begin{aligned}e^{n^2\pi^2 t}\dot{\hat{u}}_n(t) + n^2\pi^2 e^{n^2\pi^2 t}\hat{u}_n(t) &= 2(-1)^{n+1}e^{n^2\pi^2 t}\\ \int_0^{t'}\frac{d}{dt}\left[e^{n^2\pi^2 t}\hat{u}_n(t)\right]dt &= 2(-1)^{n+1}\int_0^{t'}e^{n^2\pi^2 t}\,dt\\ e^{n^2\pi^2 t'}\hat{u}_n(t') - \hat{u}_n(0) &= 2(-1)^{n+1}\left(\frac{e^{n^2\pi^2 t'}}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right)\\ \hat{u}_n(t') &= \frac{2(-1)^{n+1}}{n^2\pi^2} + \left(\hat{u}_n(0) - \frac{2(-1)^{n+1}}{n^2\pi^2}\right)e^{-n^2\pi^2 t'}.\end{aligned}$$

To calculate $\hat{u}_n(0)$, we use the integral formula we wrote down earlier, and the given initial conditions $u(x,0) = g(x) = 0$.

$$\begin{aligned}\hat{u}_n(0) &= 2\int_0^1 u(x,0)\cos(n\pi x)\,dx\\ &= 2\int_0^1 0\cdot\cos(n\pi x)\,dx = 0.\end{aligned}$$

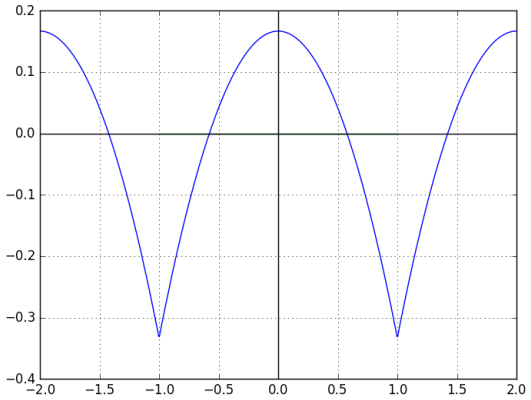
So our coefficients simplify to

$$\hat{u}_n(t) = \frac{2(-1)^{n+1}}{n^2\pi^2} - \frac{2(-1)^{n+1}}{n^2\pi^2}e^{-n^2\pi^2 t}.$$

So our solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x) - \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2\pi^2}e^{-n^2\pi^2 t}\cos(n\pi x).$$

Notice that the solution is written so that the first sum has no dependence on t. The second sum has a dependence on t, but since it is in a negative exponent, this is the transient part of the solution. It goes to zero as $t \to \infty$. So the first sum is the equilibrium solution. Here is a graph of the first 100 terms of the equilibrium solution.



Notice that on $[0, 1]$ it does appear to satisfy the boundary conditions. The first derivative at $x = 0$ appears to be 0, and the first derivative at $x = 1$ appears to be -1. This suggests that our solution is correct.

We can calculate the equilibrium solution directly from the PDE. Since $u_t = 0$ at equilibrium, the PDE simplifies to the separable ODE

$$\frac{d^2u}{dx^2} = -1,$$

which has the solution

$$u_{eq}(x) = -\frac{1}{2}x^2 + C_1x + C_2.$$

Both boundary conditions tell us that $C_1 = 0$, but tell us nothing about C_2, so we have to compute the value of C_2 using the average value of the initial conditions.

$$\begin{aligned}
\int_0^1 u(x)\,dx &= \int_0^1 \left(-\frac{1}{2}x^2 + C_2\right) dx = \int_0^1 u(x,0)\,dx \\
\int_0^1 u(x)\,dx &= -\frac{1}{6} + C_2 = \int_0^1 g(x)\,dx \\
\int_0^1 u(x)\,dx &= -\frac{1}{6} + C_2 = 0 \\
C_2 &= \frac{1}{6}.
\end{aligned}$$

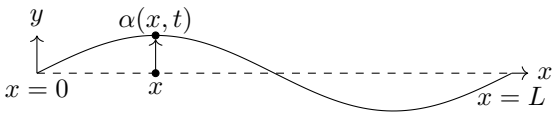
So the equilibrium solution is

$$u_{eq}(x) = -\frac{1}{2}x^2 + \frac{1}{6}.$$

If we graph this function, we see that it exactly matches the graph of the equilibrium part of the Fourier series solution on $[0, 1]$, demonstrating that at least the equilibrium part of our solution is correct.

1.7 The Wave Equation

Imagine that we have a tightly stretched wire depicted by the diagram below. At equilibrium, it is stretched from $x = 0$ to $x = L$ in a straight line (the dashed line in the diagram).



The position of a point on the wire which is at x when the wire is in its equilibrium position has position

$$\alpha(x,t) = (v(x,t), u(x,t)),$$

when displaced from its equilibrium position. The quantity $v(x,t)$ is the horizontal displacement of the point, and $u(x,t)$ is the vertical displacement of the point. We will assume that the vibrations of the wire are *small* and entirely vertical, so that $v(x,t) = x$, always, and $u(x,t)$ is small.

Note: With the heat equation, u was the function that gave the temperature as a function of position and time. Here, u gives the vertical displacement of the wire as a function of position and time.

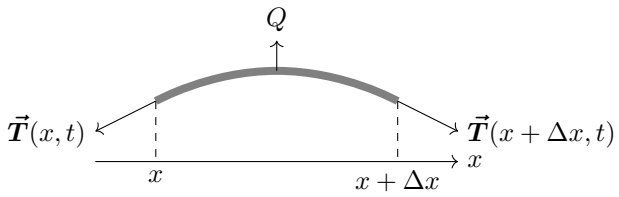
There are waves that don't obey these simplifying assumptions. They are nonlinear waves and are much harder to solve than the linear waves that we will be focused on.

We will assume the mass density $\rho(x)$ of the wire are known, so the total mass of the wire between $x = a$ and $x = b$ is found by integrating over the interval

$$m_a^b = \int_a^b \rho(x)\,dx.$$

We will also assume the wire is perfectly flexible. That is, it offers no resistance to bending. In other words, when you bend it there is no restoring force so the wire will just as easily maintain a curled position as a straight position when it is not stretched tight. The wire does, however, resist stretching with a restoring force called **tension**. The tension $T(x,t)$ will depend on how much the wire is being stretched at position x and time t. The direction of the tension force is always tangent to the wire.

There are other possible forces acting on the wire. These include gravitational force, magnetic force, electrostatic force, and so on. These forces are all external to the wire, so they can be combined into one external *net* force function $\rho(x,t)Q(x,t)$. It is actually a force density. It is in this form because the external forces are generally proportional to the mass of the wire. Since it is a force density, to calculate the force, we have to integrate from x to $x + \Delta x$ to obtain the external force on that piece of the wire.

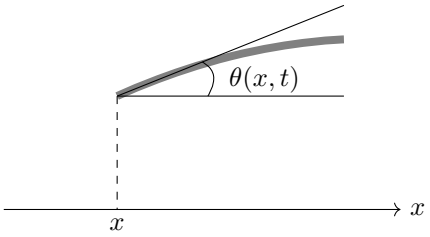


If we consider the part of the wire between x and $x + \Delta x$, the tension $\vec{\boldsymbol{T}}(x,t)$ is pulling it to the left, $\vec{\boldsymbol{T}}(x + \Delta x, t)$ is pulling it to the right, and the force

$$\int_x^{x+\Delta x} \rho(x')Q(x')\,dx',$$

is pulling it vertically. We can assume that Q is only pulling vertically. If there was a horizontal component, there would be horizontal motion which is excluded by our original assumptions.

Let $\theta(x,t)$ denote the angle that the wire makes with the horizontal at position x and time t. Positive θ indicates positive slope, and negative θ indicates negative slope.



The only forces with horizontal components are the tension forces. Since there is no horizontal motion, there is no net horizontal force, and so the horizontal components of the tension forces must sum to zero, so

$$T(x + \Delta x, t)\cos\theta(x + \Delta x, t) - T(x,t)\cos\theta(x,t) = 0.$$

We can assume that the angle is small so $\cos\theta(x,t) \approx 1$, so

$$T(x + \Delta x, t) \approx T(x,t).$$

In other words, the tension is essentially independent of x. We will also generally assume the tension is independent of time. That is, the wire isn't being tightened or loosened as the wave occurs. So T is constant. All of this is only true for relatively small waves.

The vertical components of the tension add to

$$T\sin\theta(x + \Delta x, t) - T\sin\theta(x,t).$$

For small angles, $\sin\theta \approx \theta \approx \tan\theta \approx \frac{\partial u}{\partial x}$. In other words, θ is approximately the angle of the tangent of u. So the tension sums to

$$T\frac{\partial u}{\partial x}\Big|_x^{x+\Delta x} = \int_x^{x+\Delta x} T\frac{\partial^2 u}{\partial x'^2}\,dx'.$$

Newton's second law then tells us that

$$\int_x^{x+\Delta x} \left(T\frac{\partial^2 u}{\partial x'^2} + \rho Q \right) dx' = \int_x^{x+\Delta x} \rho u_{tt}\, dx',$$

where the left side is the net force on the segment of wire, and the right side is the mass times acceleration. Since this is true for every segment of wire, it must be true that

$$\rho u_{tt} = \left(T\frac{\partial^2 u}{\partial x'^2} + \rho Q \right),$$

which we can write as the one-dimensional **wave equation**

$$u_{tt} = c^2 u_{xx} + Q,$$

where $c = \sqrt{\frac{T}{\rho}}$.

The quantity c is the **wave speed**. In a guitar string, for example, higher c means a higher note. Higher tension and lower mass makes a higher note since $c = \sqrt{\frac{T}{\rho}}$.

Most musical instruments can be explained by the wave equation. The wave equation can be used to describe the vibration of a guitar string and the sound waves such as those in flutes or organ tubes. The wave equation essentially describes any musical instrument that plays notes. It doesn't work as well to describe percussion instruments.

1.7.1 Dirichlet Boundary Conditions

With Dirichlet boundary conditions, we are given the functions

$$\begin{aligned} u(0,t) &= \alpha(t) \\ u(L,t) &= \beta(t). \end{aligned}$$

For a wire, Dirichlet boundary conditions means the vertical displacements of the ends of the wire are specified. In particular, if $\alpha = \beta = 0$, then the wires are pinned in place. This is like a guitar string, where the ends are pinned in place.

For a tube of air, Dirichlet boundary conditions means the pressure is specified at the open ends of the tube.

1.7.2 Neumann Boundary Conditions

With Neumann boundary conditions, we are given

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \alpha(t) \\ \frac{\partial u}{\partial x}(L,t) &= \beta(t). \end{aligned}$$

Recall that $\frac{\partial u}{\partial x}(x,t)$ gives the angle that the wire makes with the horizontal at position x and time t. So Neumann boundary conditions for a wire means the angles that the ends make with the horizontal are specified. Most commonly, we are given that $\alpha = \beta = 0$ which implies that no vertical force is acting on the ends of the wire. This means the ends of the wires are free to move up and down and a frictionless manner. They are never free to move horizontally.

In a tube, Neumann boundary conditions correspond to a tube with closed ends.

1.7.3 Robin Boundary Conditions

With Robin boundary conditions, we are given

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) + \alpha(0,t) &= f_0(t) \\ \frac{\partial u}{\partial x}(L,t) - \alpha(L,t) &= f_L(t). \end{aligned}$$

So neither u nor $\frac{\partial u}{\partial x}$ are specificed, but we are given a linear relationship between them.

For a wire, this corresponds to the ends of the wire being allowed to move up and down, however, there is a restoring force. Imagine that each end of the wire is attached to a spring which is attached to the equilibrium position of the end of the wire. So the ends of the wire are allowed to move up and down, but when they do, the springs pull them back toward the equilibrium position.

For a tube of air, this corresponds to the ends being closed by a flexible membrane like a balloon. That is, the air is allowed to move through the ends, but there is a restoring force.

1.7.4 Periodic Boundary Conditions

With periodic boundary conditions, we are given

$$\begin{aligned} u(-L,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t). \end{aligned}$$

For a wire or tube of air, this corresponds to the wire or tube being bent into a circle, but with tension. This is difficult to visualize with a wire, because not only is the wire bent into a circle, but throughout, there is tension in the wire.

1.7.5 Separation of Variables

In this section, we solve the homogeneous wave equation

$$u_{tt} = c^2 u_{xx},$$

using separation of variables. Instead of a single initial condition (as with the heat equation), we now need two

initial conditions, since the wave equation is second order in time.

In this case, we will solve for homogeneous Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned}$$

and we are given the initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x). \end{aligned}$$

Step 1

In our first step, we separate variables by defining

$$u(x,t) = X(x)T(t).$$

Substituting it into the PDE gives us

$$\begin{aligned} X(x)T''(t) &= c^2 X''(x)T(t) \\ \frac{X''}{X} &= \frac{T''}{c^2 T} = -\lambda. \end{aligned}$$

We use the negative sign with our constant λ so that the eigenvalues will end up being positive.

Our separated ODEs are

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + c^2 \lambda T &= 0. \end{aligned}$$

Step 2

The x direction is the homogeneous direction, so the eigenvalue problem which we solve with the boundary conditions is

$$X'' + \lambda X = 0.$$

From earlier, we know the solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Step 3

In the third step, we solve the time equation, using the eigenvalues from the previous step.

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0,$$

where $\left(\frac{n\pi c}{L}\right) > 0$. The solution is

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right).$$

Notice that we now have two linearly independent solutions for the time equation. This is because it is a second order ODE where for the heat equation, it was a first order ODE.

Step 4

Combining the solutions, we get

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Step 5

In the final step, we use the initial conditions to compute the coefficients. From the first initial conditions equation, plugging $t = 0$ into the solution found in the previous step, we get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

This is just the sine series for $f(x)$, and we know how to calculate the coefficients for sine series

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The other equation for the initial conditions is the first time derivative of u. Differentiating the solution found in step 4 with respect to time gives us

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) + B_n \left(\frac{n\pi c}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right).$$

The solution $u(x,t)$ is a Fourier series with respect to x. We know this because the eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$ are functions of x. We can always take the derivative of a Fourier series with respect to another variable, in this case t.

Evaluating the time derivative at $(x,0)$ gives us

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

This is another sine series, and the coefficients are given by

$$\begin{aligned} B_n \frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ B_n &= \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

We are now done. We found the general solution, and we have computed the two coefficients.

Notice that the initial condition $f(x) = u(x,0)$ is just the initial displacement of every point on the wire and $g(x) = u_t(x,0)$ is the initial velocity of every point on the wire.

Using the Cartesian-polar transformation equations and the angle sum formula, we can write

$$A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) = R_n \sin\left(\frac{n\pi ct}{L} + \theta_n\right),$$

Where

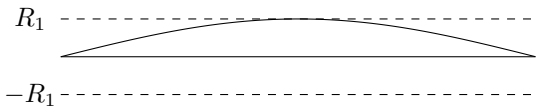
$$\begin{aligned} \text{amplitude} &= R_n = \sqrt{A_n^2 + B_n^2} \\ \text{phase shift} &= \theta_n = \tan^{-1}\left(\frac{A_n}{B_n}\right) \\ \text{period} &= \frac{2L}{cn} \\ \text{frequency} &= \frac{cn}{2L}. \end{aligned}$$

The frequency $\frac{c}{2L}$ with $n = 1$ is called the **fundamental frequency** or **first harmonic**. The higher freqencies with $n = 2, 3, \ldots$ are called **overtones**.

This allows us to rewrite our solution as

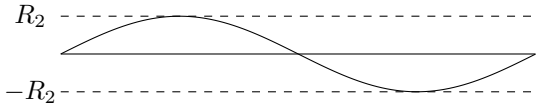
$$u(x,t) = \sum_{n=1}^{\infty} R_n \sin\left(\frac{n\pi ct}{L} + \theta_n\right) \sin\left(\frac{n\pi x}{L}\right).$$

If we plot the $n = 1$ term, we get something that looks like this:



This is the graph of $\sin\left(\frac{\pi x}{L}\right)$ with amplitude $R_1 \sin\left(\frac{\pi ct}{L} + \theta_1\right)$. Notice that the amplitude varies with time. So this is a wire vibrating up and down between R_1 and $-R_1$ with frequency $\frac{c}{2L}$.

If we plot the $n = 2$ term, we get something that looks like:



This is the graph of $\sin\left(\frac{2\pi x}{L}\right)$ with amplitude $R_2 \sin\left(\frac{2\pi ct}{L} + \theta_2\right)$. Notice that the amplitude varies with time. So this is a wire vibrating up and down between R_2 and $-R_2$ with frequency $\frac{2c}{2L}$. This is a wire vibrating twice is fast as the $n = 1$ case. In music terms, it is one octave higher.

The entire solution is the superposition of an infinite number of these frequencies–each one given different weight.

1.7.6 Summary

Solve the wave equation like you would the heat equation. Use separation of variables for the homogeneous case. For the nonhomogeneous wave equation, use eigenfunction expansion.

1.8 Sturm-Liouville Eigenvalue Problems

1.8.1 Introduction

So far, the only eigenvalue problem, we've had to solve, and which has appeared repeatedly was of the form

$$X'' + \lambda X = 0.$$

In this section, we look at more general eigenvalue problems called **Sturm-Liouville** eigenvalue problems. They have the form

$$\frac{d}{dx}\left(p\frac{d}{dx}\phi\right) + q\phi + \lambda\sigma\phi = 0.$$

The eigenfunctions are $\phi(x)$, and the eigenvalues λ are the unknowns. The variables p, q, and σ are known functions of x on $a \le x \le b$. All eigenvalue problems that come up, in practice, are of this form.

Notice that if $p = 1$, $q = 0$, and $\sigma = 1$, we get an eigenvalue problem of the familiar form

$$\phi'' + \lambda\phi = 0.$$

If $p = x$, $q = 0$, and $\sigma = \frac{1}{x}$ we get one of the form

$$x^2\phi'' + x\phi' + \lambda\phi = 0.$$

This eigenvalue problem leads to Bessel functions.

In general, we will assume that p, q, and σ are nice functions. This means $p(x), \sigma(x) > 0$ for $a \le x \le b$ and $p(x)$, $q(x)$, and $\sigma(x)$ are continuous for $a \le x \le b$. If these conditions are met, the problem is *regular*. Otherwise, the problem is *singular*.

The boundary conditions need to be linear and **self-adjoint** (or symmetric).

For a regular eigenvalue problem with "nice" boundary conditions, the following theorems hold true:

1. The eigenvalues are all real.
2. There are infinitely many eigenvalues, but we can order them as $\lambda_1 < \lambda_2 < \dots$ and $\lambda_n \to \infty$ as $n \to \infty$.
3. Each eigenfunction $\phi_n(x)$ is real-valued and has $n - 1$ zeros on $a < x < b$. Zeros at the endpoints aren't counted.
4. The eigenfunctions are "complete". That is, we can write any reasonable function $f(x)$ as
$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$
5. The eigenfunctions are orthogonal with respect to $\sigma(x)$. That is
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0, \quad \text{if } \lambda_m \neq \lambda_n,$$
so
$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$
6. We have the **Rayleigh quotients**
$$\lambda = \frac{-p\phi\phi'\Big|_{x=a}^{b} f\int_a^b \left(p\phi'^2 - q\phi^2\right)dx}{\int_a^b \phi^2\sigma\,dx}.$$

1.8.2 Derivation

We begin with the eigenvalue problem

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\phi(x)\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0,$$

on $a \le x \le b$ with Dirichlet, Neumann, Robin, or periodic boundary conditions. Since it is an eigenvalue problem, the boundary conditions are necessarily homogeneous meaning the zero function satisfies them.

If the eigenvalue problem is **regular**, then $p(x), \sigma(x) > 0$ on $a \le x \le b$ and $p(x)$, $q(x)$, and $\sigma(x)$ are continuous on $a \le x \le b$.

To write things more compactly, we use the linear operator

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

so the eigenvalue equation given above can be written as

$$L\phi + \lambda\sigma\phi = 0.$$

We look now at the **commutator**, $uLv - vLu$,

$$\begin{aligned} uLv - vLu &= u\left[(pv')' + qv\right] - v\left[(pu')' + qu\right] \\ &= u\,(pv')' - v\,(pu')'. \end{aligned}$$

Adding and subtracting $u'pv'$ gives us

$$uLv - vLu = u\,(pv')' + u'pv' - v\,(pu')' - v'pu'.$$

This is now the result of two product rules, so we can write

$$\begin{aligned} uLv - vLu &= (upv')' - (vpu')' \\ &= (p[uv' - vu'])'. \end{aligned}$$

This gives us **Lagrange's identity**

$$uLv - vLu = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right].$$

Lagrange's identity is useful because it allows us to integrate $uLv - vLu$ which gives us **Green's identity**

$$\int_a^b (uLv - vLu)\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x=a}^{b}.$$

This relates the eigenvalue equation (via the left side of Green's identity) and its boundary conditions (via the right side of Green's identity).

We say that the operator L is **self-adjoint** (or symmetric) with respect to our boundary conditions if for any functions u and v satisfying the boundary conditions, it is true that

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x=a}^{b} = 0.$$

Recall that Dirichlet boundary conditions are

$$\begin{aligned}\phi(a) &= 0\\ \phi(b) &= 0.\end{aligned}$$

If two functions u and v satisfy the Dirichlet boundary conditions, that is $u(a) = v(a) = 0$ and $u(b) = v(b) = 0$, then

$$\begin{aligned}p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{a}^{b} &= p(b)\left(u(b)v'(b) - v(b)u'(b)\right)\\ &\quad -p(a)\left(u(a)v'(a) - v(a)u'(a)\right)\\ &= p(b)\left(0\cdot v'(b) - 0\cdot u'(b)\right)\\ &\quad -p(a)\left(0\cdot v'(a) - 0\cdot u'(a)\right)\\ &= 0.\end{aligned}$$

Therefore, L is self-adjoint with respect to Dirichlet boundary conditions.

If we work it out with Neumann, Robin, or periodic boundary conditions, we get 0 for each of them. So L is self-adjoint with respect to all of these types of boundary conditions.

Note, periodic boundary conditions are defined a little differently now since we include $p(x)$.

$$\begin{aligned}\phi(a) &= \phi(b)\\ p(a)\phi'(a) &= p(b)\phi'(b).\end{aligned}$$

Previously, $p(a) = p(b)$ was just equal to 1, so we were able to neglect those. Regardless, L is still self-adjoint with respect to periodic boundary conditions.

As long as your operator is self-adjoint with respect to your boundary conditions, your PDE will fit very well into the standard framework, and shouldn't be too difficult to solve. If you don't have self-adjointness, the problem becomes much harder to solve.

From here on, we will work with the eigenvalue equation of the form

$$L\phi + \lambda\sigma\phi = 0,$$

with some boundary conditions, and we will assume that L is self-adjoint with respect to the boundary conditions. We know that it definitely will be self-adjoint if the boundary conditions are Dirichlet, Neumann, Robin, or periodic.

First, we will show that our eigenfunctions are orthogonal. Suppose we have two distinct eigenvalues λ_1 and λ_2, and we have corresponding eigenfunctions ϕ_1 and ϕ_2. We want to show that ϕ_1 and ϕ_2 are orthogonal. That is, we want to show that

$$\int_a^b \phi_1(x)\phi_2(x)\sigma(x)\,dx = 0.$$

Note, in earlier cases when checking orthogonality we did not include $\sigma(x)$ since it was 1. Multiplying the left and right side by $\lambda_1 - \lambda_2$, gives us

$$\begin{aligned}(\lambda_1 - \lambda_2)\int_a^b \phi_1\phi_2\sigma\,dx &= \int_a^b \left[\phi_2(\lambda_1\sigma\phi_1) - \phi_1(\lambda_2\sigma\phi_2)\right]dx\\ &= \int_a^b \left[-\phi_2 L\phi_1 + \phi_1 L\phi_2\right]dx\\ &= p\left(\phi_1\phi_2' - \phi_2\phi_1'\right)\Big|_a^b\\ &= 0.\end{aligned}$$

To go from line one to line two, we use the equality $L\phi + \lambda\sigma\phi = 0$. In the next, we use Green's identity. To get to the final line, we use the fact that ϕ_1 and ϕ_2 are eigenfunctions and therefore they satisfy the boundary conditions and by the self-adjointness assumption. Since our result is zero, this tells us that ϕ_1 and ϕ_2 are orthogonal eigenfunctions.

Next, we want to show that the eigenvalues are real. Suppose we have an eigenvalue λ with eigenfunction ϕ. This means that $L\phi + \lambda\sigma\phi = 0$ and ϕ satisfies the boundary conditions. Taking the complex conjugates gives us $L\overline{\phi} + \overline{\lambda}\sigma\overline{\phi} = 0$ and $\overline{\phi}$ satisfies the boundary conditions. Since p, q, and σ are real functions, their complex conjugates equal themselves. Using pretty much the same process as above, we get that

$$\begin{aligned}(\lambda - \overline{\lambda})\int_a^b \phi\overline{\phi}\sigma\,dx &= \int_a^b \left[\overline{\phi}(\lambda\sigma\phi) - \phi(\overline{\lambda}\sigma\overline{\phi})\right]dx\\ &= \int_a^b \left[-\overline{\phi}L\phi + \phi L\overline{\phi}\right]dx\\ &= p\left(\phi\overline{\phi}' - \overline{\phi}\phi'\right)\Big|_a^b\\ &= 0.\end{aligned}$$

However, a function times its complex conjugate gives its modulus squared,

$$\int_a^b \phi\overline{\phi}\sigma\,dx = \int_a^b |\phi|^2\sigma\,dx > 0.$$

Since $|\phi|^2 > 0$ and σ is assumed positive, it must be that $\lambda - \overline{\lambda} = 0$. That is, $\lambda = \overline{\lambda}$ which implies that eigenvalues λ are real. We can also assume that our eigenfunctions ϕ are real.

It is often possible to show that all eigenvalues are single. Suppose that ϕ_1 and ϕ_2 are both eigenfunctions for the eigenvalue λ. Then

$$\begin{aligned}L\phi_1 + \lambda\sigma\phi_1 &= 0\\ L\phi_2 + \lambda\sigma\phi_2 &= 0,\end{aligned}$$

and ϕ_1 and ϕ_2 satisfy the boundary conditions. Multiply one of them by ϕ_1 and the other by ϕ_2 and subtract them to get

$$\begin{aligned} 0 &= \phi_2(L\phi_1 + \lambda\sigma\phi_1) - \phi_1(L\phi_2 + \lambda\sigma\phi_2) \\ &= \phi_2 L\phi_1 - \phi_1 L\phi_2 \\ &= \frac{d}{dx}\left[p\left(\phi_2\phi_1' - \phi_1\phi_2'\right)\right], \end{aligned}$$

by Lagrange's identity. This implies that

$$p\left(\phi_2\phi_1' - \phi_1\phi_2'\right) = C_1,$$

is a constant since the derivative is zero. If the boundary conditions tell us that $C_1 = 0$, which they will if they are Dirichlet, Neumann, or Robin boundary conditions, then we are in business. We know that $p \neq 0$ given the starting assumption of regularity, so it must be the case that

$$\phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x) = 0,$$

for all x. Notice that this is the numerator of the quotient rule, so we can write it as

$$\frac{d}{dx}\left(\frac{\phi_2}{\phi_1}\right) = 0,$$

which implies that

$$\frac{\phi_2}{\phi_1} = C_2,$$

or $\phi_2 = C_2\phi_1$. This is what we wanted to show. If one eigenfunction is the same as another eigenfunction times a constant, then they are the same eigenfunction. What we have shown is that if the boundary conditions are Dirichlet, Neumann, or Robin, for example, then the eigenvalues are all single. That is, every eigenvalue has a unique eigenfunction that is associated with it. This is especially helpful when trying to find the eigenvalues numerically.

1.8.3 Rayleigh Quotients

If we start with the general eigenvalue problem

$$\frac{d}{dx}\left(p\frac{d}{dx}\phi\right) + q\phi + \lambda\sigma\phi = 0,$$

then multiply both sides by ϕ and integrate from a to b, we get

$$\begin{aligned} \int_a^b \phi\left[\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi\right] dx &= 0 \\ \int_a^b \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\phi\,dx + \int_a^b q\phi^2\,dx + \lambda\int_a^b \phi^2\sigma\,dx &= 0. \end{aligned}$$

Finally, we solve for λ to get the Rayleigh quotient

$$\lambda = \frac{-\int_a^b \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\phi\,dx - \int_a^b q\phi^2\,dx}{\int_a^b \phi^2\sigma\,dx}.$$

We can make this more compact by using the operator L as

$$\lambda = \frac{-\int_a^b \phi L\phi\,dx}{\int_a^b \phi^2\sigma\,dx}. \tag{1.6}$$

Alternatively, we can do an integration by parts

$$\int_a^b \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\phi\,dx = p\phi\frac{d\phi}{dx}\Big|_{x=a}^b - \int_a^b p\left(\frac{d\phi}{dx}\right)^2 dx,$$

to write the Rayleigh quotient in a different way

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_{x=a}^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right] dx}{\int_a^b \phi^2\sigma\,dx}. \tag{1.7}$$

Notice that if $-p\phi\frac{d\phi}{dx}\Big|_{x=a}^b \geq 0$ and $q(x) \leq 0$, then every term in Eq. (1.7) is positive and so $\lambda > 1$. In actual physical situations, $q(x)$ is almost always negative. Just knowing the sign of the eigenvalues can often tell you a lot about how the solution will behave. For example, it can often tell you if there will be exponential growth or decay or if there will be steady behavior.

If we can't find the eigenvalues of a PDE analytically, we can use Rayleigh quotients to approximate them numerically.

For any function u, we write the Rayleigh quotient of u in one of two ways

$$\begin{aligned} RQ[u] &= \frac{-\int_a^b uLu\,dx}{\int_a^b u^2\sigma\,dx} \\ RQ[u] &= \frac{-puu'\Big|_a^b + \int_a^b \left[p\left(u'\right)^2 - qu^2\right] dx}{\int_a^b u^2\sigma\,dx}. \end{aligned}$$

Next, we expand u as

$$u = \sum_{n=1}^{\infty} \hat{u}_n\phi_n.$$

Multiplying by ϕ_m and σ and integrating gives us

$$\int_a^b u\phi_m\sigma\,dx = \sum_{n=1}^{\infty} \hat{u}_n \int_a^b \phi_n\phi_m\sigma\,dx.$$

By orthogonality, the right side has only one nonzero term, which occurs when $n = m$, so

$$\int_a^b u\phi_m\sigma\,dx = \hat{u}_m\int_a^b \phi_m^2\sigma\,dx.$$

Solving for the coefficients of u gives us

$$\hat{u}_m = \frac{\int_a^b u\phi_m\sigma\,dx}{\int_a^b \phi_m^2\sigma\,dx}.$$

If u satisfies the boundary conditions, we can apply the L operator term-by-term to get

$$\begin{aligned} Lu &= \sum_{n=1}^{\infty} \hat{u}_n L\phi_n \\ &= -\sum_{n=1}^{\infty} \hat{u}_n\lambda_n\sigma\phi_n, \end{aligned}$$

since $L\phi + \lambda\sigma\phi = 0$.

Suppose that

$$Lu = \sum_{n=1}^{\infty} \hat{v}_n \sigma \phi_n,$$

and u satisfies the boundary conditions. Then multiplying both sides by ϕ_n and integrating from a to b gives us

$$\int_a^b (Lu)\phi_n \, dx = \sum_{n=1}^{\infty} \hat{v}_n \int_a^b \sigma \phi_n \phi_m \, dx.$$

By orthogonality, the right side has only one nonzero term—when $n = m$, so

$$\int_a^b (Lu)\phi_n \, dx = \hat{v}_m \int_a^b \phi_m^2 \sigma \, dx.$$

Recall that **Green's identity**

$$\int_a^b (\phi_m Lu - uL\phi_m) \, dx = p\left(\phi_m \frac{du}{dx} - u\frac{d\phi_m}{dx}\right)\Big|_a^b,$$

simplifies to

$$\int_a^b (\phi_m Lu - uL\phi_m) \, dx = 0,$$

if u satisfies the boundary conditions. This implies that

$$\int_a^b (Lu)\phi_m \, dx = \int_a^b uL\phi_m \, dx.$$

That is, it allows us to move the L operator from u to ϕ_m. Since $L\phi + \lambda\sigma\phi = 0$, this gives us

$$\int_a^b (Lu)\phi_m \, dx = -\lambda_m \int_a^b u\phi_m \sigma \, dx.$$

So we can solve for $\hat{v}_m$ to get

$$\begin{aligned} \hat{v}_m &= \frac{-\lambda_m \int_a^b u\phi_m \sigma \, dx}{\int_a^b \phi_m^2 \sigma \, dx} \\ &= -\lambda_m \hat{u}_m. \end{aligned}$$

1.8.4 Rayleigh's Method

We have the eigenvalue problem

$$L\phi + \lambda\sigma\phi = 0,$$

with some boundary conditions. We have some trial function

$$u(x) = \sum_{n=1}^{\infty} \hat{u}_n \phi_n(x),$$

which gives us

$$\hat{u}_n = \frac{\int_a^b u\phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx}.$$

If $u(x)$ satisfies the boundary conditions, then

$$Lu(x) = \sum_{n=1}^{\infty} \hat{u}_n L\phi_n(x) = -\sum_{n=1}^{\infty} \lambda_n \hat{u}_n \sigma(x)\phi_n(x),$$

and the Rayleigh quotient can be expressed in the two ways

$$\begin{aligned} RQ[u] &= \frac{-\int_a^b uLu \, dx}{\int_a^b u^2 \sigma \, dx} \\ RQ[u] &= \frac{-puu'\Big|_a^b + \int_a^b \left[p\,(u')^2 - qu^2\right] dx}{\int_a^b u^2 \sigma \, dx}. \end{aligned}$$

Note that the Rayleigh quotient of an eigenfunction yields the corresponding eigenvalue. That is,

$$RQ[\phi_n] = \lambda_n.$$

It is true that no matter what u you choose, provided that it satisfies the boundary conditions, that

$$RQ[u] \geq \lambda_1.$$

That is, the Rayleight quotient of u is greater than or equal to the smallest eigenvalue. To verify this, we begin with the numerator of the Rayleigh quotient and substitute in the series definitions for u and Lu

$$\begin{aligned} -\int_a^b uLu \, dx &= -\int_a^b \left(\sum_{n=1}^{\infty} \hat{u}_n \phi_n\right)\left(-\sum_{m=1}^{\infty} \lambda_m \hat{u}_m \sigma \phi_m\right) dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_m \hat{u}_n \hat{u}_m \int_a^b \phi_n \phi_m \sigma \, dx. \end{aligned}$$

By orthogonality, all terms in the integral on the right are zero except when $n = m$, so

$$-\int_a^b uLu\,dx = \sum_{n=1}^{\infty} \lambda_n \hat{u}_n^2 \int_a^b \phi_n^2 \sigma\,dx.$$

This result is now the numerator of $RQ[u]$. Doing the denominator in the same manner, we get

$$\begin{aligned}\int_a^b u^2\sigma\,dx &= \int_a^b \left(\sum_{n=1}^{\infty} \hat{u}_n\phi_n\right)\left(\sum_{m=1}^{\infty} \hat{u}_m\phi_m\right)\sigma\,dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \hat{u}_n\hat{u}_m \int_a^b \phi_n\phi_m\sigma\,dx \\ &= \sum_{n=1}^{\infty} \hat{u}_n^2 \int_a^b \phi_n^2\sigma\,dx.\end{aligned}$$

Putting the numerator and denominator together, we get

$$RQ[u] = \frac{\sum_{n=1}^{\infty} \lambda_n \hat{u}_n^2 \int_a^b \phi_n^2\sigma\,dx}{\sum_{n=1}^{\infty} \hat{u}_n^2 \int_a^b \phi_n^2\sigma\,dx}.$$

Since everything is squared except for σ which is presumed by regularity to be positive, the right side is positive and larger than or equal to the smallest eigenvalue. That is,

$$RQ[u] \geq \frac{\sum_{n=1}^{\infty} \lambda_1 \hat{u}_n^2 \int_a^b \phi_n^2\sigma\,dx}{\sum_{n=1}^{\infty} \hat{u}_n^2 \int_a^b \phi_n^2\sigma\,dx} = \lambda_1.$$

Given this, we can approximate λ_1 of ϕ by using a convenient choice of u that satisfies the same boundary conditions. This is called **Rayleigh's method**.

Example

Recall the eigenvalue problem

$$X'' + \lambda X = 0,$$

with the Dirichlet boundary conditions

$$\begin{aligned}X(0) &= 0 \\ X(\pi) &= 0.\end{aligned}$$

This is really the Sturm-Liouville eigenvalue problem

$$LX = X'', \qquad 0 \leq x \leq \pi,$$

with $\sigma, p = 1$ and $q = 0$. We know the solution to this eigenvalue problem is

$$X_n(x) = \sin(nx), \qquad \lambda_n = n^2.$$

In particular, the smallest eigenvalue and its eigenfunction are

$$\lambda_1 = 1, \qquad X_1(x) = \sin x.$$

Recall that the Rayleigh quotient of u can be expressed as

$$RQ[u] = \frac{-puu'\Big|_a^b + \int_a^b \left[p\,(u')^2 - qu^2\right] dx}{\int_a^b u^2\sigma\,dx}.$$

In this case $q = 0$ and $p = \sigma = 1$. Also, note that if u satisfies the boundary conditions then the first term in the numerator of Eq. (1.7) is zero, and the rayleigh quotient in our case simplifies to

$$RQ[u] = \frac{\int_0^\pi (u')^2\,dx}{\int_0^\pi u^2\,dx}.$$

We know that $RQ[\phi_n] = \lambda_n$. To confirm that we haven't made a mistake, we'll check this in our case. We have that $\phi_1 = \sin x$, so

$$\begin{aligned}RQ[\sin x] &= \frac{\int_0^\pi \cos^2 x\,dx}{\int_0^\pi \sin^2 x\,dx} \\ &= \frac{\int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx}{\int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx} \\ &= \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1 = \lambda_1,\end{aligned}$$

so it checks out in our case.

To estimate λ_1 using the Rayleigh method, we need a function u that satisfies the boundary conditions. A possibility is a polynomial that is zero at $x = 0$ and $x = \pi$, so we will use

$$v(x) = x(\pi - x).$$

Plugging this into the Rayleigh quotient gives us

$$\begin{aligned}RQ[x(\pi - x)] &= \frac{\int_0^\pi (\pi - 2x)^2\,dx}{\int_0^\pi (\pi x - x^2)^2\,dx} \\ &= \frac{\frac{\pi^3}{3}}{\frac{\pi^5}{30}} = \frac{10}{\pi^2} \approx 1.013211.\end{aligned}$$

So our estimation of λ_1 is only off by about a percent, which is a very good approximation for a quick computation by hand.

Another standard choice for u is a piecewise linear (i.e. triangular) function. In our case, we could let

$$w(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

The Rayleigh quotient is then

$$\begin{aligned}RQ[w] &= \frac{\int_0^\pi dx}{\int_0^{\frac{\pi}{2}} x^2\,dx + \int_{\frac{\pi}{2}}^\pi (\pi - x)^2\,dx} \\ &= \frac{\pi}{\frac{\pi^3}{12}} = \frac{12}{\pi^2} \approx 1.2159.\end{aligned}$$

So in this case, our approximation is off by about 20%.

In general, to compute a better approximation of λ_1, the standard method is to divide the interval (in our case $0 \leq x \leq \pi$) into many segments, then use a linear combination of triangular functions using arbitrary values a_k. Then the Rayleigh quotient becomes a function of the arbitrary constants a_k. Now, RQ can be minimized to get a better estimation of λ_1. This is the basic idea behind the **finite elements method**.

1.8.5 Rayleigh-Ritz Method

To approximate higher eigenvalues such as $\lambda_2, \lambda_3, \ldots$, there is little you can do by hand. Instead, we can use the Rayleigh-Ritz method to perform a computational estimation.

To estimate the first k eigenvalues, you have to use k different functions. Then the first ones will be good estimations, but the last ones won't be. For example, to get a good estimation of the first several eigenvalues, you might use 10 functions. Then the first two eigenvalues will be fairly well approximated, but the last eight will not be very good.

Here's is an illustration of the Rayleigh-Ritz method using two functions v and w. We start with a matrix A of numerator-like stuff

$$A = \begin{bmatrix} -\int_a^b vLv\,dx & -\int_a^b wLv\,dx \\ -\int_a^b vLw\,dx & -\int_a^b wLw\,dx \end{bmatrix}.$$

We have a second matrix containing the denominator-like stuff

$$B = \begin{bmatrix} \int_a^b v^2\sigma\,dx & \int_a^b vw\sigma\,dx \\ \int_a^b wv\sigma\,dx & \int_a^b w^2\sigma\,dx \end{bmatrix}.$$

Then to get our approximations of λ_1 and λ_2, we solve

$$\det(A - \lambda B) = 0.$$

Example

Using the same eigenvalue problem and functions v and w used in the previous example, the Rayleigh-Ritz method gives us

$$A = \begin{bmatrix} \frac{1}{3}\pi^3 & \frac{1}{2}\pi^2 \\ \frac{1}{2}\pi^2 & \pi \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{30}\pi^5 & \frac{5}{96}\pi^4 \\ \frac{5}{96}\pi^4 & \frac{1}{12}\pi^3 \end{bmatrix}.$$

Then solving $\det(A - \lambda B) = 0$, we get

$$\lambda = \frac{1}{\pi^3}\left(\frac{208}{3} \pm \frac{32\sqrt{31}}{3}\right),$$

or

$$\begin{aligned} \lambda_1 &\approx 1.0075 \\ \lambda_2 &\approx 13.04. \end{aligned}$$

Notice that this gives us a much better approximation of λ_1 than the Rayleigh method, but the approximation of λ_2 is not good at all. The actual values of the eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 4. \end{aligned}$$

To get really good estimations of the first 10,000 eigenvalues, we would, for example, perform this process on a computer using a million functions.

1.8.6 Least Squares Approximation

If we want to find an eigenfunction expansion for a particular function such as

$$u(x) = \sum_{n=1}^{\infty} \hat{u}_n \phi_n(x),$$

then we have the integral formula

$$\hat{u}_n = \frac{\int_a^b u(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx},$$

to find the coefficients. However, we can only use the integral formula if we know $u(x)$ and $\phi_n(x)$. How do we find the coefficients if we don't know these functions? It is typically very hard to find all the coefficients precisely. Instead, we do approximations and truncations. Instead of finding the full series, we truncate it at some point. We want

$$u(x) \approx \sum_{n=1}^{N} \alpha_n \phi_n(x),$$

for some coefficients α_n. We want to find the α_n's that make the approximation as good as possible given our choice of N.

To find a good approximation, we need a way to measure how good an approximation is. To do that, we measure in terms of **mean squared error**, which is defined as

$$\text{MSE} = \int_a^b \left(u(x) - \sum_{n=1}^{N} \alpha_n \phi_n(x)\right)^2 \sigma(x)\,dx. \qquad (1.8)$$

We want to find the α_n's that minimize this error. In some sense, the MSE is a function of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$. Expanding the integral by squaring the integrand, gives us

$$\begin{aligned}
\text{MSE} &= \int_a^b u^2(x)\sigma(x)\,dx - 2\int_a^b u(x)\left(\sum_{n=1}^N \alpha_n\phi_n(x)\right)\sigma(x)\,dx + \int_a^b \left(\sum_{n=1}^N \alpha_n\phi_n(x)\right)\left(\sum_{m=1}^N \alpha_m\phi_m(x)\right)\sigma(x)\,dx \\
&= \int_a^b u^2(x)\sigma(x)\,dx - 2\sum_{n=1}^N \alpha_n \int_a^b u(x)\phi_n(x)\sigma(x)\,dx + \sum_{n=1}^N\sum_{m=1}^N \alpha_n\alpha_m \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx \\
&= \int_a^b u^2(x)\sigma(x)\,dx - 2\sum_{n=1}^N \alpha_n \int_a^b u(x)\phi_n(x)\sigma(x)\,dx + \sum_{n=1}^N \alpha_n^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx.
\end{aligned}$$

In the last line, we simplified using the orthogonality condition that the third integral is zero unless $n = m$. Notice that in terms of α_n, our equation for the MSE is a quadratic. The integrals just evaluate to numbers. We'll use this fact to *complete the square* thereby simplifying it further. To reduce the clutter, we let

$$\begin{aligned}
A_n &= \int_a^b \phi_n^2(x)\sigma(x)\,dx \\
B_n &= \int_a^b u(x)\phi_n(x)\sigma(x)\,dx \\
C &= \int_a^b u^2(x)\sigma(x)\,dx,
\end{aligned}$$

then our equation becomes

$$\begin{aligned}
\text{MSE} &= C - 2\sum_{n=1}^N B_n\alpha_n + \sum_{n=1}^N A_n\alpha_n^2 \\
&= C + \sum_{n=1}^N \left(A_n\alpha_n^2 - 2B_n\alpha_n\right) \\
&= C + \sum_{n=1}^N A_n\left(\alpha_n^2 - 2\frac{B_n}{A_n}\alpha_n\right) \\
&= C + \sum_{n=1}^N A_n\left(\left[\alpha_n - \frac{B_n}{A_n}\right]^2 - \frac{B_n^2}{A_n^2}\right) \\
&= C + \sum_{n=1}^N A_n\left[\alpha_n - \frac{B_n}{A_n}\right]^2 - \sum_{n=1}^N \frac{B_n^2}{A_n}.
\end{aligned}$$

The only term that we can control is the middle one since the others are fixed for us. To minimize the MSE, we make this term as small as possible. Since it is squared, it is as small as possible when it is equal to zero. This occurs when

$$\alpha_n = \frac{B_n}{A_n} = \frac{\int_a^b u(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} = \hat{u}_n.$$

So provided that the ϕ_n's are orthogonal, the MSE is minimized when we choose $\alpha_n = \hat{u}_n$, and it is equal to

$$\begin{aligned}
\text{MSE} &= C - \sum_{n=1}^N A_n\frac{B_n^2}{A_n^2} \\
&= C - \sum_{n=1}^N A_n\alpha_n^2 \\
&= \int_a^b u^2(x)\sigma(x)\,dx - \sum_{n=1}^N \alpha_n^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx.
\end{aligned}$$

Recall from Eq. (1.8) that MSE > 0 since the quantity in the integrand is squared. This implies that

$$\sum_{n=1}^N \alpha_n^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx \le \int_a^b u^2(x)\sigma(x)\,dx.$$

Since the α_n do not depend on N, we can let $N \to \infty$ and the inequality still holds. That is,

$$\sum_{n=1}^\infty \alpha_n^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx \le \int_a^b u^2(x)\sigma(x)\,dx.$$

This result is called **Bessel's inequality**.

It is very common to get equality (instead of inequality) in Bessel's inequality. In particular, if the ϕ_n are the eigenfunctions from a Sturm-Liouville problem, the equality

$$\sum_{n=1}^\infty \alpha_n^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx = \int_a^b u^2(x)\sigma(x)\,dx,$$

holds and is called **Parseval's equation**.

Geometrically, the least-squares approximation is just the projection of u into the infinite-dimensional space of the ϕ_n. This makes it by far the most convenient and most used error estimate.

The least squares estimate will miss a sharp spike in u unless N is very large. A **uniform best fit** won't miss such a spike, and would perhaps be a better error estimate since it would make the worst point error (i.e. find the point at which the error is greatest and minimize it) as small as possible. However, the uniform best fit is more difficult to implement and can take a lot more work.

1.8.7 Summary

A **Sturm-Liouville** eigenvalue problem has the form

$$\frac{d}{dx}\left(p\frac{d}{dx}\phi\right) + q\phi + \lambda\sigma\phi = 0.$$

The eigenfunctions are $\phi(x)$, and the eigenvalues λ are the unknowns. The variables p, q, and σ are known functions of x on $a \le x \le b$. All eigenvalue problems that come up, in practice, are of this form.

In general, we will assume that p, q, and σ are nice functions. This means $p(x), \sigma(x) > 0$ for $a \le x \le b$ and $p(x)$, $q(x)$, and $\sigma(x)$ are continuous for $a \le x \le b$. If these conditions are met, the problem is *regular*. Otherwise, the problem is *singular*.

For Sturm-Liouville eigenvalue problems, the eigenvalues and eigenfunctions are real. The eigenfunctions are *complete*, so any reasonable function $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

The eigenfunctions are orthogonal with respect to $\sigma(x)$. That is

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0, \quad \text{if } \lambda_m \neq \lambda_n,$$

so

$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$

If we start with the general eigenvalue problem

$$\frac{d}{dx}\left(p\frac{d}{dx}\phi\right) + q\phi + \lambda\sigma\phi = 0,$$

then multiply both sides by ϕ and integrate from a to b, we get

$$\begin{aligned}\int_a^b \phi\left[\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi\right]dx &= 0\\ \int_a^b \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\phi\,dx + \int_a^b q\phi^2\,dx + \lambda\int_a^b \phi^2\sigma\,dx &= 0.\end{aligned}$$

Finally, we solve for λ to get the Rayleigh quotient

$$\lambda = \frac{-\int_a^b \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\phi\,dx - \int_a^b q\phi^2\,dx}{\int_a^b \phi^2\sigma\,dx}.$$

Integrating by parts gives us

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_{x=a}^{b} + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}.$$

Notice that if $-p\phi\frac{d\phi}{dx}\Big|_{x=a}^{b} \ge 0$ and $q(x) \le 0$, then every term in the equation above is positive and so $\lambda > 1$.

The Rayleigh quotient of an eigenfunction gives the associated eigenvalue. That is,

$$RQ[\phi_n] = \lambda_n.$$

We can approximate the first eigenvalue by choosing a nice function $u(x)$ that satisfies the boundary conditions and taking its Rayleigh quotient. That is,

$$RQ[u] = \frac{-puu'\Big|_a^b + \int_a^b \left[p\,(u')^2 - qu^2\right]dx}{\int_a^b u^2\sigma\,dx} \ge \lambda_1.$$

If $f(x)$ on $[a,b]$, is defined by the Fourier series

$$f(x) = \sum_{n=1}^{\infty} \hat{f}_n \phi_n(x),$$

where

$$\hat{f}_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx},$$

then **Bessel's inequality** tells us that

$$\sum_{n=1}^{\infty} |\hat{f}_n|^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx \le \int_a^b |f(x)|^2\sigma(x)\,dx.$$

In the case of equality, the above is called **Parseval's equation**.

Given a Fourier series, to "apply" Parseval's equation means to take the absolute value of both sides, square both sides, then integrate both sides over the interval of definition. For example, given the Fourier sine series for a function defined on $[0, L]$

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

applying Parseval's equation means to simplify the following

$$\int_0^L |f(x)|^2\,dx = \sum_{n=0}^{\infty} |b_n|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx.$$

1.9 Higher Dimensional PDEs

1.9.1 Vibrations in a Drumhead

For a vibrating drumhead, we have the wave equation

$$u_{tt} = c^2 \Delta u,$$

in a circular region D. The boundary condition is $u = 0$ on the boundary ∂D, and u and u_t are given at $t = 0$ as initial conditions. This is our 2-dimensional wave equation with Dirichlet boundary conditions.

Step 1: Separate Variables

In the first step, we separate variables as in previous examples. We have two space dimensions and one time dimension, so our solution will be a function of all three of them. That is, $u = u(x, y, t)$.

To separate variables, we separate the space and time dimensions as

$$u(x, y, t) = \phi(x, y) \cdot T(t).$$

Plugging this into the wave equation given above, gives us

$$\begin{aligned} \phi T'' &= c^2 (\Delta \phi) T \\ \frac{T''}{c^2 T} &= \frac{\Delta \phi}{\phi} = -\lambda. \end{aligned}$$

Notice that once again, we end up with a function of space on one side and a function of time on the other side, which implies that both sides are constant. In this case, we choose the constant to be $-\lambda$. This gives us the pair of ODEs

$$\begin{aligned} T'' + c^2 \lambda T &= 0 \\ \Delta \phi + \lambda \phi &= 0. \end{aligned}$$

Step 2: Separate the E.V. Equation

In step two, we solve the eigenvalue equation which is the spatial equation

$$\begin{aligned} \Delta \phi + \lambda \phi &= 0 \qquad \text{in } D \\ \phi &= 0 \qquad \text{on } \partial D. \end{aligned}$$

To solve this, we need to specify the region D. We let

$$D = \{x^2 + y^2 < a^2\}.$$

That is, D is a disk of radius a. Since it's a circular region, we will be using polar coordinates. Recall that in polar coordinates

$$\Delta \phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta},$$

so our eigenvalue PDE becomes

$$\begin{aligned} \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \lambda \phi &= 0 \\ \phi(a, \theta) &= 0. \end{aligned}$$

We start by separating the variables using

$$\phi = R(r) \cdot \Theta(\theta).$$

Plugging this into the PDE gives us

$$\begin{aligned} R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r}R\Theta'' + \lambda R\Theta &= 0 \\ \frac{R''}{R} + \frac{R''}{rR} + \frac{\Theta''}{r^2\Theta} + \lambda &= 0 \\ \frac{r^2 R''}{R} + \frac{rR''}{R} + r^2\lambda &= -\frac{\Theta''}{\Theta} = \gamma. \end{aligned}$$

So we get the pair of ODEs

$$\begin{aligned} \Theta'' + \gamma \Theta &= 0 \\ R'' + \frac{1}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right) R &= 0. \end{aligned}$$

For the first ODE, we have periodic boundary conditions on $-\pi \le \theta \le \pi$, and for the second one, we have $R(a) = 0$, and that $R(0)$ is finite.

We know the solution to the θ equation since it is our "standard" eigenvalue problem

$$\gamma_n = n^2, \quad \Theta_n(\theta) = \cos(n\theta), \sin(n\theta), \ n = 0, 1, 1, 2, 2, \ldots$$

Instead of the eigenfunctions $\cos(n\theta)$ and $\sin(n\theta)$, we could also use $e^{in\theta}$ and $e^{-in\theta}$ if it would be more convenient.

Step 3: Solve R-equation

The R-equation is now

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right) R = 0,$$

with $R(a) = 0$ and $R(0) =$ finite.

To solve the R-equation, we first have to show that λ is positive. To do that, we look at the Rayleigh quotient. Recall that our eigenvalue equation is $\Delta\phi + \lambda\phi = 0$. Multiplying both sides by ϕ and integrating over the region gives us

$$\iint_D \left(\phi \Delta \phi + \lambda \phi^2\right) dx\, dy = 0.$$

Solving for λ gives us the Rayleigh quotient

$$\lambda = \frac{-\iint_D \phi \Delta \phi \, dx\, dy}{\iint_D \phi^2 \, dx\, dy}.$$

Previously, we rewrote the Rayleigh quotient by using integration by parts. Here, we cannot do that since our integral is over a two-dimensional region.

Integration by parts, which is basically the integral form of the product rule, is a one-dimensional operation. We need the 2-dimensional analogue of integration by parts. To find it, we use the **divergence theorem**

$$\iint_D \operatorname{div}\vec{\boldsymbol{F}}\,dx\,dy = \int_{\partial D} \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{n}}\,dS.$$

The divergence theorem is basically the higher dimensional analogue of the fundamental theorem of calculus.

If g is a scalar valued function and $\vec{\boldsymbol{F}}$ is a two-dimensional vector function, then

$$\begin{aligned}
\operatorname{div}(g\vec{\boldsymbol{F}}) &= \operatorname{div}\left(\langle gF_1, gF_2\rangle\right)\\
&= \frac{\partial}{\partial x}(gF_1) + \frac{\partial}{\partial y}(gF_2)\\
&= \frac{\partial g}{\partial x}F_1 + g\frac{\partial F_1}{\partial x} + \frac{\partial g}{\partial y}F_2 + g\frac{\partial F_2}{\partial y}\\
&= \left(\frac{\partial g}{\partial x}F_1 + \frac{\partial g}{\partial y}F_2\right) + g\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)\\
&= \operatorname{grad} g\cdot\vec{\boldsymbol{F}} + g\,\operatorname{div}(\vec{\boldsymbol{F}}).
\end{aligned}$$

Now suppose that $\vec{\boldsymbol{F}} = \operatorname{grad} f$, then

$$\begin{aligned}
\operatorname{div}(g\,\operatorname{grad} f) &= \operatorname{grad} g\cdot\,\operatorname{grad} f + g\,\operatorname{div}(\,\operatorname{grad} f)\\
&= \operatorname{grad} g\cdot\,\operatorname{grad} f + g\Delta f\\
&= \nabla g\nabla f + g\Delta f.
\end{aligned}$$

In particular, if $f = g$ then

$$\operatorname{div}(g\nabla g) = \nabla g\nabla g + g\Delta g.$$

This tells us that

$$\iint_D \operatorname{div}(g\nabla g)\,dx\,dy = \iint_D \nabla g\nabla g\,dx\,dy + \iint_D g\Delta g\,dx\,dy,$$

but the divergence theorem tells us that

$$\iint_D \operatorname{div}(g\nabla g)\,dx\,dy = \int_{\partial D} g\nabla g\cdot\vec{\boldsymbol{n}}\,dS.$$

Equating the two, rearranging, and replacing g with ϕ gives us

$$\iint_D \phi\Delta\phi\,dx\,dy = \int_{\partial D}\phi\nabla\phi\cdot\vec{\boldsymbol{n}}\,dS - \iint_D \nabla\phi\nabla\phi\,dx\,dy.$$

Notice that this is the higher dimensional analogue of integration by parts. Instead of integrating a function times its derivative, we are integrating a function times its Laplacian. Instead of evaluating the first part of the result at the endpoints (i.e. the one-dimensional boundary), we are now evaluating it along a two-dimensional boundary, hence the line integral along ∂D.

Using this analogue of integration by parts, we can rewrite our Rayleigh quotient as

$$\lambda = \frac{-\int_{\partial D}\phi\nabla\phi\cdot\vec{\boldsymbol{n}}\,dS + \iint_D \nabla\phi\nabla\phi\,dx\,dy}{\iint_D \phi^2\,dx\,dy}.$$

Since $\phi = 0$ along the boundary, the first integral in the numerator is zero, so

$$\lambda = \frac{\iint_D \nabla\phi\nabla\phi\,dx\,dy}{\iint_D \phi^2\,dx\,dy}.$$

The numerator and denominator are both positive which implies that $\lambda > 0$.

Now that we have shown that $\lambda > 0$, we are ready to return to our R-equation.

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right)R = 0,$$

with $R(a) = 0$ and $R(0) =$ finite. Note that R is a function of r. Since $\lambda > 0$, we can now make the substitution $\rho = \sqrt{\lambda}r$, then by the chain rule $\frac{dR}{dr} = \sqrt{\lambda}\frac{dR}{d\rho}$ and $\frac{d^2R}{dr^2} = \lambda\frac{d^2R}{d\rho^2}$. Plugging these in gives us

$$\lambda R_{\rho\rho} + \frac{\sqrt{\lambda}}{\rho}\sqrt{\lambda}R_\rho + \left(\lambda - n^2\frac{\lambda}{\rho^2}\right)R = 0.$$

Since $\lambda \neq 0$, we can cancel it from every term to get

$$R_{\rho\rho} + \frac{1}{\rho}R_\rho + \left(1 - \frac{n^2}{\rho^2}\right)R = 0. \tag{1.9}$$

Now that we got rid of λ, the equation is now in the form called **Bessel's equation of order** n which is a form that we can solve. The solution cannot be written in terms of elementary functions, but is well known and called the **Bessel function of the first kind of order** n, and it is denoted by $J_n(\rho)$. Our solution to the R-equation will be $J_n(\sqrt{\lambda}r)$, and our eigenfunctions will look like $J_n(\sqrt{\lambda}r)\cos(n\theta)$ and $J_n(\sqrt{\lambda}r)\sin(n\theta)$. Now we need to find the J_n's and the λ_n's.

Equation (1.9) is not in Sturm-Liouville form, but we can transform it into that form.

We look for a series solution of the form

$$R(\rho) = \rho^\alpha\sum_{k=0}^{\infty} a_k\rho^k, \qquad a_0 \neq 0.$$

Our series is centered at $\rho = 0$. Since that is a singular point (in the denominator of the ODE), we need the ρ^α out front. Simplifying and taking the partial derivatives of the series gives us

$$\begin{aligned}
R(\rho) &= \sum_{k=0}^{\infty} a_k\rho^{k+\alpha}\\
R_\rho &= \sum_{k=0}^{\infty} a_k(k+\alpha)\rho^{k+\alpha-1}\\
R_{\rho\rho} &= \sum_{k=0}^{\infty} a_k(k+\alpha)(k+\alpha-1)\rho^{k+\alpha-2}.
\end{aligned}$$

Plugging these into Eq. (1.9) and simplifying gives us

$$0 = \sum_{k=0}^{\infty}\left[(k+\alpha)^2 - n^2\right]a_k\rho^{k+\alpha-2} + \sum_{k=0}^{\infty} a_k\rho^{k+\alpha}.$$

Shifting the index of the second sum by letting $k = l - 2$ gives us

$$0 = \sum_{k=0}^{\infty} \left[(k+\alpha)^2 - n^2\right] a_k \rho^{k+\alpha-2} + \sum_{l=2}^{\infty} a_{l-2}\rho^{l+\alpha-2}.$$

Since l is just a dummy variable, we can combine the sums after pulling the first two terms out of the first sum

$$\begin{aligned} 0 &= (\alpha^2 - n^2)a_0\rho^{\alpha-2} + ((\alpha+1)^2 - n^2)a_1\rho^{\alpha-1} \\ &\quad + \sum_{k=2}^{\infty} \left[((k+\alpha)^2 - n^2)a_k + a_{k-2}\right] \rho^{k+\alpha-2}. \end{aligned}$$

Since it is equal to zero, all the coefficients must be zero. That is

$$(\alpha^2 - n^2)a_0 = 0 \implies \alpha = \pm n,$$

since we arranged that $a_0 \neq 0$. The $\alpha = -n$ case is the solution that blows up at the origin, called the Bessel function of the second kind. In our case, $\alpha \neq -n$ since one of our boundary conditions is that $R(0)$ is finite. We will only consider the $\alpha = n$ case.

Also, we have that

$$((\alpha+1)^2 - n^2)a_1 = 0, \implies a_1 = 0,$$

since $\alpha = n$ and $n + 1 \neq n$. Finally, equating the coefficient in the sum to zero gives us

$$\begin{aligned} ((k+\alpha)^2 - n^2)a_k + a_{k-2} &= 0 \\ ((k+n)^2 - n^2)a_k + a_{k-2} &= 0 \\ a_k &= \frac{-a_{k-2}}{k(k+2n)}, \end{aligned}$$

for $k = 2, 3, 4, \ldots$ So we have a recursion relationship between a coefficient and an earlier coefficient. Note that a_0 is arbitrary and $a_1 = 0$, all the coefficients where n is odd will be zero, since each coefficient depends on the one two coefficients earlier. Computing the first several nonzero ones, we get

$$\begin{aligned} a_2 &= \frac{-a_0}{2 \cdot 2(n+1)} \\ a_4 &= \frac{a_0}{4 \cdot 2 \cdot 2 \cdot 2(n+1)(n+2)} \\ a_6 &= \frac{-a_0}{6 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2(n+1)(n+2)(n+3)}. \end{aligned}$$

In general,

$$a_{2j} = \frac{(-1)^j a_0 n!}{j! 2^{2j} (n+j)!}.$$

Our series is then

$$\sum_{j=0}^{\infty} \frac{(-1)^j a_0 n!}{j! 2^{2j} (n+j)!} \rho^{n+2j}.$$

Note that n is a constant here. We can choose the arbitrary first coefficient to be

$$a_0 = \frac{1}{2^n n!},$$

in order to simplify the series, and what we get is the standard version of **Bessel's function of the first kind**

$$J_n(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{\rho}{2}\right)^{n+2j}.$$

This is like the Taylor series of $J_n(\rho)$. Some properties include

- It has an infinite radius of convergence
- If n is even then $J_n(\rho)$ is even
- If n is odd then $J_n(\rho)$ is odd
- n doesn't have to be an integer. If it is not, we can use the gamma function instead of the factorial function

$$J_n(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(n+j+1)} \left(\frac{\rho}{2}\right)^{n+2j}.$$

Asymptotically, when ρ is large

$$J_n(\rho) \approx \sqrt{\frac{2}{\pi\rho}} \cos\left(\rho - \frac{\pi}{4} - \frac{n\pi}{2}\right) + O\left(\rho^{-\frac{3}{2}}\right).$$

So asymptotically, the Bessel function of the first kind looks like a shifted and decaying cosine function.

Returning to our ODE Eq. (1.9), we have that the solution is

$$R(r) = J_n(\rho) = J_n(\sqrt{\lambda} r).$$

To satisfy the boundary conditions, we need that $R(0)$ is finite, which means $J_n(0)$ is finite and $R(a) = 0$, which means $J_n(\sqrt{\lambda} a) = 0$. This means we need $\sqrt{\lambda} a = z_{nm}$ where z_{nm} is the mth positive zero of J_n. So

$$\lambda_{nm} = \left(\frac{z_{nm}}{a}\right)^2.$$

Recall that the eigenvalues in the one-dimensional case had the form $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Now that we're dealing with the two-dimensional case, the lambda require two indices. Basically, the number of indices matches the number of spatial dimensions in the problem. Instead of a length L in the denominator, we have a radius a. Instead of having $n\pi$, the nth zero of sine in the numerator, we now have the mth zero of the Bessel function in the numerator.

Our eigenfunctions will now have the form

$$J_n(\sqrt{\lambda_{nm}} r)\cos(n\theta), \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots$$

Note, the cosine could also be a sine.

Note that

$$\int_{-\pi}^{\pi}\int_0^a J_n(\sqrt{\lambda_{nm}}r)\cos(n\theta)J_{n'}(\sqrt{\lambda_{n'm'}}r)\cos(n'\theta)\,r\;dr\,d\theta = 0,$$

when $m \neq n$. That is, we have orthogonality. Also, if $m \neq m'$, then

$$\begin{aligned}\int_{-\pi}^{\pi}\int_0^a J_n(\sqrt{\lambda_{nm}}r)\cos(n\theta)J_n(\sqrt{\lambda_{nm'}}r)\cos(n\theta)\,r\;dr\,d\theta &= 0\\ \int_{-\pi}^{\pi}\cos^2 n\theta\,d\theta\int_0^a J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nm'}}r)\,r\;dr &= 0.\end{aligned}$$

Since the cosine integral is not zero, the other one must be. So our orthogonality condition for Bessel functions is that if $m \neq m'$, then

$$\int_0^a J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nm'}}r)\,r\;dr = 0.$$

The value of

$$\int_0^a J_n^2(\sqrt{\lambda_{nm}}r)\,r\,dr,$$

must be approximated numerically or looked up in a table. We need these integrals for the coefficients of the series.

Step 4: Solve T-equation

The T equation

$$T'' + c^2\lambda T = 0,$$

has solution $\cos(c\sqrt{\lambda}t)$ and $\sin(c\sqrt{\lambda}t)$.

Step 5: Combine

Since everything is multiplied together, the final solution to the wave equation on a 2D disk looks very messy.

We need to remember the ODE that the Bessel function comes from.

$$\begin{aligned}u(\vec{\boldsymbol{x}},t) &= \sum_{m=1}^{\infty}\left[A_{0m}J_0(\sqrt{\lambda_{0m}}r)\cos(c\sqrt{\lambda_{0m}}t) + C_{0m}J_0(\sqrt{\lambda_{0m}}r)\sin(c\sqrt{\lambda_{0m}}t)\right]\\ &\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\cos n\theta\cos\left(c\sqrt{\lambda_{nm}}t\right)\\ &\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\sin n\theta\cos\left(c\sqrt{\lambda_{nm}}t\right)\\ &\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\cos n\theta\sin\left(c\sqrt{\lambda_{nm}}t\right)\\ &\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\sin n\theta\sin\left(c\sqrt{\lambda_{nm}}t\right).\end{aligned}$$

1.9.2 Heat Flow in a Disk

Consider the circularly symmetric case of heat flow in a disk.

For heat flow in a disk, we have the heat equation

$$u_t = k\Delta u,$$

in a circular region $D = \{x^2 + y^2 < a^2\}$. The boundary condition is $u = 0$ on the boundary $x^2 + y^2 = a^2$ when $t \geq 0$. The initial condition is $u = B$ when $x^2 + y^2 < a^2$ and when $t = 0$.

Step 1: Separate Variables

In the first step, we separate variables as in previous examples. Since we are dealing with a disk, we will work in polar coordinates. We have two space dimensions and one time dimension, however, since we are considering the circularly symmetric case, the solution will not depend on θ, it will only depend on r. So we can immediately separate the PDE into functions of r and t.

In the circularly symmetric case, the Laplacian is

$$\Delta u = \frac{1}{r}u_r + u_{rr},$$

so we can rewrite the PDE as

$$u_t = k\left(\frac{1}{r}u_r + u_{rr}\right).$$

To separate variables, we separate the space and time dimensions as

$$u(r,t) = R(r)\cdot T(t).$$

Plugging this into the heat equation given above, gives us

$$\begin{aligned} RT' &= k\frac{1}{r}R'T + kR''T \\ \frac{T'}{kT} &= \frac{R'}{rR} + \frac{R''}{R} = -\lambda. \end{aligned}$$

Notice that once again, we end up with a function of space on one side and a function of time on the other side, which implies that both sides are constant. In this case, we choose the constant to be $-\lambda$. This gives us the pair of ODEs

$$\begin{aligned} T' + k\lambda T &= 0 \\ R'' + \frac{1}{r}R' + \lambda R &= 0. \end{aligned}$$

Step 2: Solve the E.V. Equation

Now we solve the eigenvalue equation which is the spatial equation

$$R'' + \frac{1}{r}R' + \lambda R = 0,$$

with $R(a) = 0$ and $R(0) =$ finite.

Notice that this is a Bessel equation of the first kind of order zero, so its solutions (and our eigenfunctions) are the Bessel functions

$$R(r) = J_0(\sqrt{\lambda}r).$$

and given $J_0(\sqrt{\lambda}a) = 0$ the eigenvalues are

$$\lambda_n = \left(\frac{z_n}{a}\right)^2,$$

where $n = 1, 2, 3, \ldots$

Step 3: Solve T-equation

The T equation

$$T' + k\lambda T = 0,$$

has solution

$$T = e^{-k\lambda t}.$$

Step 4: Combine

Combining the eigenfunctions and the time solutions gives us the general solution

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}r)e^{-k\lambda_n t}.$$

Step 5: Initial Conditions

Since $u = B$ when $t = 0$, we get

$$B = u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}r).$$

This Bessel function has the orthogonality condition

$$\int_0^a J_0(\sqrt{\lambda_n}r)J_0(\sqrt{\lambda_m}r)r\,dr = 0,$$

when $n \neq m$, so we multiply both sides by $J_0(\sqrt{\lambda_m}r)r$ and integrate to get

$$\int_0^a BJ_0(\sqrt{\lambda_m}r)r\,dr = \sum_{n=1}^{\infty} A_n \int_0^a J_0(\sqrt{\lambda_n}r)J_0(\sqrt{\lambda_m}r)r\,dr.$$

On the right, by orthogonality, the integral will be zero except when $n = m$, so this simplifies to

$$\int_0^a BJ_0(\sqrt{\lambda_m}r)r\,dr = A_m \int_0^a J_0^2(\sqrt{\lambda_m}r)\,dr.$$

Solving for A_m and replacing m's by n's gives us the coefficients

$$A_n = \frac{\int_0^a BJ_0(\sqrt{\lambda_n}r)r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_n}r)\,dr}.$$

Our final solution is then

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}r)e^{-k\lambda_n t},$$

where

$$A_n = \frac{\int_0^a BJ_0(\sqrt{\lambda_n}r)r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_n}r)\,dr}.$$

1.9.3 The Legendre Equation

Before starting the next section, we will solve the Legendre equation, the ordinary second order differential equation

$$((1-z^2)u')' + \gamma u = 0,$$

where u is a function of z and the prime symbols denote differentiation. This is a Sturm-Liouville problem, but it is not regular since p is not greater than zero at the boundaries.

We look for series solutions of the form

$$u(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Substituting this and its derivative into the ODE, simplifying, and reindexing, gives us

$$0 = \sum_{k=0}^{\infty} \left[(k+1)(k+2)a_{k+2} + (\gamma - (k+1)k)a_k\right] z^k.$$

This implies that

$$a_{k+2} = \frac{k(k+1)-\gamma}{(k+1)(k+2)} a_k, \qquad k = 0, 1, 2, \ldots$$

The fraction limits to 1 for large k, so for large k, $a_{k+2} \approx a_k$. In other words, the coefficients a_k are roughly constant for large k, and approximate a geometric series.

We need our solution to have finite values at $z = \pm 1$, but the only way we can have finite solutions is if the a_k are zero for large k. By looking at the formula for a_{k+2} given above, we see that this implies that γ must be of the form $l(l+1)$, for some $l = 0, 1, 2, \ldots$ in which case all coeffcients a_{l+2} and onward will be zero. This gives us the **Legendre polynomials**.

The first several Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z \\ P_2(z) &= \frac{1}{2}(3z^2 - 1) \\ P_3(z) &= \frac{1}{2}(5z^3 - 3z) \\ P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3) \\ P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Some properties of the Legendre polynomials are

- Orthogonality on $-1 \le z \le 1$:

$$\int_1^{-1} P_l(z) P_m(z)\, dz = 0.$$

- $P_l(z)$ has degree l
- If l is even then $P_l(z)$ is even
- If l is odd then $P_l(z)$ is odd
- A closed form expression for them is

$$P_l(z) = \frac{1}{2^l} \sum_{j=0}^{\lfloor l/2 \rfloor} \frac{(-1)^j}{j!} \frac{(2l-2j)!}{(l-2j)(l-j)!} z^{l-2j}.$$

The Legendre polynomials also satisfy various important equations. The important thing is to be able to recognize the first few Legendre polynomials and to know that the Legendre equation has been completely solved. We can always look up the properties in a reference text.

1.9.4 Vibrations in a Solid Ball

We will now solve the PDE

$$\Delta X + \lambda X = 0,$$

on the sphere $x^2 + y^2 + z^2 \le a^2$ with Dirichlet boundary conditions $0 \le r \le a$, $-\pi \le \theta \le \pi$, and $0 \le \phi \le \pi$. Our transformation equations are

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi. \end{aligned}$$

The Laplacian in spherical coordinates is

$$\Delta X = X_{rr} + \frac{2}{r} X_r + \frac{1}{r^2} \left[\frac{1}{\sin^2\phi} X_{\theta\theta} + \frac{1}{\sin\phi} (\sin\phi X_\phi)_\phi \right].$$

Substituting this into the PDE and then making the separation replacement $X = R(r) \cdot Y(\theta, \phi)$ and simplifying, gives us the PDE

$$0 = \lambda r^2 + \frac{r^2 R_{rr} + 2r R_r}{R} + \frac{\frac{1}{\sin^2\phi} Y_{\theta\theta} + \frac{1}{\sin\phi} (\sin\phi Y_\phi)_\phi}{Y}.$$

Notice that the first two terms depend only on r and the third term depends only on the angles. Since the whole thing equals zero, the third term must be the negative of the sum of the first two terms. That is, for some constant γ, we have that

$$\begin{aligned} \gamma &= \lambda r^2 + \frac{r^2 R_{rr} + 2r R_r}{R} \\ -\gamma &= \frac{\frac{1}{\sin^2\phi} Y_{\theta\theta} + \frac{1}{\sin\phi} (\sin\phi Y_\phi)_\phi}{Y}. \end{aligned}$$

We now have two separated equations. The R equation is

$$R_{rr} + \frac{2}{r} R_r + \left(\lambda - \frac{\gamma}{r^2} \right) R = 0,$$

where $R(a) = 0$ and $R(0) =$ finite. The Y-equation is

$$\frac{1}{\sin^2\phi} Y_{\theta\theta} + \frac{1}{\sin\phi} (\sin\phi Y_\phi)_\phi + \gamma Y = 0,$$

where θ is periodic with period 2π and Y is finite at $\phi = 0, \pi$.

The Angular Part

The angular part of our PDE is given by

$$\frac{1}{\sin^2\phi} Y_{\theta\theta} + \frac{1}{\sin\phi} (\sin\phi Y_\phi)_\phi + \gamma Y = 0,$$

where θ is periodic with period 2π and Y is finite at $\phi = 0, \pi$.

The solutions to the Y-equation are called **spherical harmonics**.

To solve the Y-equation, we start by separating variables again. We let $Y(\theta, \phi) = \Theta(\theta) \cdot \Phi(\phi)$. Plugging this in, dividing by $\Theta\Phi$ and multiplying by $\sin^2\phi$ gives us

$$\frac{\Theta_{\theta\theta}}{\Theta} + \frac{\sin\phi\, (\sin\phi\ \Phi_\phi)_\phi}{\Phi} + \gamma \sin^2\phi = 0.$$

Notice that the first term depends only on θ and the rest depends only on ϕ. Since the whole thing sums to zero, the first term must be the negative of the sum of the rest.

We can relate each part of this to a constant α which gives us the separated pair of equations

$$\begin{aligned} \Theta_{\theta\theta} + \alpha\Theta &= 0 \\ \frac{1}{\sin\phi}\left(\sin\phi\ \Phi_\phi\right)_\phi + \left(\gamma - \frac{\alpha}{\sin^2\phi}\right)\Phi &= 0. \end{aligned}$$

The first is an ODE that we have solved numerous times. Its solution is

$$\Theta_m(\theta) = \cos m\theta,\ \sin m\theta, \qquad m = 0, 1, 1, 2, 2, \ldots$$

with $\alpha = m^2$. Alternatively, we can write the solution in the form

$$\Theta_m(\theta) = e^{im\theta}, \qquad m = \ldots, -2, -1, 0, 1, 2, \ldots.$$

The second ODE is of Sturm-Liouville form. We solve it by first changing variables with $s = \cos\phi$. Then $\sin^2\phi = 1 - \cos^2\phi = 1 - s^2$. Since $0 \le \phi \le \pi$, we have that $-1 \le s \le 1$. Note by the chain rule that

$$\frac{d}{d\phi} = \frac{d}{ds}\cdot\frac{ds}{d\phi} = -\sin\phi\frac{d}{ds},$$

so $\Phi_\phi = -\sin\phi\ \Phi_s$ and $\frac{1}{\sin\phi}\left(\sin\phi\ \Phi_\phi\right)_\phi = ((1-s^2)\Phi_s)_s$. So the Φ-equation becomes

$$((1-s^2)\Phi_s)_s + \left(\gamma - \frac{m^2}{1-s^2}\right)\Phi = 0,$$

where Φ is finite at $s = \pm 1$. This equation is called the **associated Legendre equation**, and its solutions are

$$\Phi(\phi) = P_l^m(\cos\phi),$$

where

$$P_l^m(z) = (1-z^2)^{\frac{m}{2}}\frac{d^m}{dz^m}\left(P_l(z)\right),$$

are the **associated Legendre polynomials** and $P_l(z)$ are the Legendre polynomials. Because of the mth derivative in the equation above, m must be less than l.

Our solutions $\Phi(\phi)$ are polynomials in sines and cosine, and they give the shapes of the electron shells for the hydrogen atom.

The solution to the angular part

$$\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}\left(\sin\phi Y_\phi\right)_\phi + \gamma Y = 0,$$

of our original PDE is obtained by combining the solutions to the Θ-equation and the Φ-equation to get

$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\phi)e^{im\theta},$$

where $\gamma = l(l+1)$, $l = 0, 1, 2, \ldots$, $m = -l, \ldots, l$, and $P_l^{|m|}$ are the associated Legendre functions.

The Radial Part

The radial part of our PDE is given by

$$R_{rr} + \frac{2}{r}R_r + \left(\lambda - \frac{l(l+1)}{r^2}\right)R = 0,$$

where $R(a) = 0$ and $R(0) =$ finite. We now know that $\gamma = l(l+1)$.

Right now we have $\frac{2}{r}$ in front of our R_r term, and we want to put it in the form of a Bessel equation which has $\frac{1}{r}$ in front of that term. To do that, we begin by making a common change of variables $w(r) = r^{-k}R(r)$ or $R = r^k w$. Taking the appropriate derivatives (remembering to use the product rule), plugging them into the R-equation and simplifying, gives us

$$w_{rr} + \frac{2(k+1)}{r}w_r + \left(\lambda + \frac{2k + k(k-1) - l(l+1)}{r^2}\right)w = 0.$$

For this to be a Bessel equation, the coefficient on w_r must be $\frac{1}{r}$. Since k is arbitrary, we choose $k = -\frac{1}{2}$, then the ODE simplifies to the Bessel equation

$$w_{rr} + \frac{1}{r}w_r + \left(\lambda - \frac{l(l+1) + \frac{1}{4}}{r^2}\right)w = 0.$$

Recall that the Bessel's equation of order n has the form

$$w_{rr} + \frac{1}{r}w_r + \left(\lambda - \frac{n^2}{r^2}\right)w = 0.$$

So we have $n^2 = l(l+1) + \frac{1}{4} = \left(l + \frac{1}{2}\right)^2$. So the solution to the w-equation is

$$w(r) = J_{l+\frac{1}{2}}(\sqrt{\lambda_n}r),$$

where

$$\lambda = \left(\frac{z}{a}\right)^2,$$

and z is a zero of $J_{l+\frac{1}{2}}$. So our solution to the R-equation is

$$R_{ln}(r) = \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda_n}r)}{\sqrt{r}},$$

where

$$\lambda_{ln} = \left(\frac{z_{ln}}{a}\right)^2,$$

z_{ln} is the nth positive zero, and $n = 1, 2, 3, \ldots$. The subscripts l and n on R indicate that R depends on the l and which zero is chosen.

The Solution

So the solution to the eigenvalue problem in 3D given at the beginning of this section is obtained by combining the angular and radial solutions

$$X_{lmn}(r,\theta,\phi) = \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda_n}r)}{\sqrt{r}}P_l^{|m|}(\cos\phi)e^{im\theta},$$

where $n = 1, 2, 3, \ldots$, $l = 0, 1, 2, \ldots$, and $m = -l, \ldots, l$. Notice that the solution has a distinct r part, a distinct θ part, and a distinct ϕ part. The r part has to do with the inside of the ball. The ϕ and θ parts have to do with the surface.

If your equations only have to do with the surface of a sphere (e.g. climate models on the surface of Earth), the equation will only have the ϕ and θ parts. When dealing with a solid sphere (e.g. stars or the inside of Earth), then you also need the r part.

1.9.5 Bohr Model of a Hydrogen Atom

The equation used to model a hydrogen atom is the Schrodinger equation

$$iu_t = -\frac{1}{2}\Delta u - \frac{1}{r}u.$$

The part $iu_t = -\frac{1}{2}\Delta u$ describes the motion of an electron in the absence of the proton, and the term $-\frac{1}{r}u$ is the potential term due to the proton. Overall, u describes where the electron is located. For this model, there is only one hydrogen atom in the whole universe, and it has one proton and one electron.

The function $u(\vec{\boldsymbol{x}}, t)$ is a complex function. If you take its modulus and square it, the result is the probability that the electron is near $\vec{\boldsymbol{x}}$ if you measure its position at time t.

To begin, we separate the space and time variables by letting $u = X(x) \cdot T(t)$. Plugging it in and simplifying gives us

$$2i\frac{T'}{T} = \frac{-\Delta X - \frac{2}{r}X}{X} = \lambda.$$

The time equation

$$T' - \frac{\lambda}{2i}T = 0,$$

has solutions

$$T = e^{\frac{\lambda}{2}it},$$

which describe oscillations or wave-like behavior. The space equation is

$$-\Delta X - \frac{2}{r}X = \lambda X.$$

We need boundary conditions for this eigenvalue equation, but in this case we don't seem to have any. The electron could be found anywhere with nonzero probability. We'll assume the boundary condition that it decays at infinity and also that $\iiint |X(\vec{\boldsymbol{x}})|^2 d\vec{\boldsymbol{x}} < \infty$ since the probability must be finite.

Spherically-symmetric Solutions

We begin by looking only for spherically-symmetric solutions to keep things simple at first. That is, we will look for solutions that depend only on r, by letting

$$X = R(r).$$

Plugging this into the eigenvalue equation gives us

$$-R_{rr} - \frac{2}{r}R_r - \frac{2}{r}R = \lambda R, \tag{1.10}$$

with the boundary conditions

$$\int_0^\infty |R(r)|^2 r^2\, dr < \infty, \quad \text{and} \quad |R(0)| < \infty.$$

How will the solutions to this PDE behave near infinity? When r is large, the middle two terms become negligible since each is divided by r. Therefore, for large r,

$$-R_{rr} \approx \lambda R.$$

In order for this new PDE to have a solution, it needs to decay at infinity to satisfy the condition $\int_0^\infty |R(r)|^2 r^2\, dr < \infty$. So λ must be negative in order to get a solution that is exponentially decaying. If $\beta = \sqrt{-\lambda}$, we get the solution

$$R(r) \approx Ce^{-\beta r}.$$

Normally, we would also include $C_2 e^{\beta r}$, but that solution doesn't decay as it needs to.

The actual solution to Eq. (1.10) will have some function instead of a constant in front of the exponential. This suggets the reasonable change of variables

$$R = w(r)e^{-\beta r}.$$

Taking the first and second derivatives of this and plugging it back into Eq. (1.10) and simplifying gives us

$$\frac{1}{2}rw_{rr} - \beta r w_r + w_r + (1-\beta)w = 0.$$

This PDE is related to **Laguerre's equation**.

Whenever a solution to a PDE isn't obvious, try a series solution. We do so here by letting

$$w(r) = \sum_{k=0}^{\infty} a_k r^k.$$

Taking the relevant partial derivatives and plugging them into the PDE above gives us

$$0 = \sum_{k=1}^{\infty} \left[\frac{k(k+1)}{2}a_k + (1-\beta k)a_{k-1}\right] r^{k-1},$$

which implies that

$$a_k = \frac{2(\beta k - 1)}{k(k+1)}a_{k-1}, \quad k = 1, 2, 3, \ldots.$$

For large k,

$$a_k \approx \frac{2\beta}{k}a_{k-1} \approx C\frac{(2\beta)^k}{k!}.$$

The series is then

$$\begin{aligned} w(r) &\approx C\sum_{k=0}^{\infty}\frac{(2\beta)^k}{k!}r^k \\ &= C\sum_{k=0}^{\infty}\frac{(2\beta r)^k}{k!} \\ &= Ce^{2\beta r}. \end{aligned}$$

But that means that $R = Ce^{2\beta r}e^{-\beta r} = Ce^{\beta r}$. This is not an acceptable solution since it doesn't decay. Therefore, for large k, the series must go to zero. That is, $a_k = 0$ for large k. If $\beta = \frac{1}{n}$ for some natural number n, then $a_n = 0$ (which you can see from the recursion formula for a_k above), then $a_{n+1} = a_{n-2} = \cdots = 0$. That is, all the terms after some nth term are zero. Then this soluton works.

The solutions $w(r)$ are the **Laguerre polynomials**.

So if X depends only on r, then the solutions are of the form

$$X = e^{-\frac{r}{n}}L_n(r), \quad n = 1, 2, 3, \ldots$$

where $L_n(r)$ is the nth Laguerre polynomial, and the eigenvalues are

$$\lambda_n = -\frac{1}{n^2}.$$

The first several ones are shown in the table below along with the eigenvalues and the full spherically-symmetric solutions $R(r)$.

n	λ	$w(r)$	$R(r)$
1	-1	1	e^{-r}
2	$-\frac{1}{4}$	$1-\frac{1}{2}r$	$e^{-\frac{1}{2}r}(1-2r)$
3	$-\frac{1}{9}$	$1-\frac{2}{3}r+\frac{2}{27}r^2$	$\vdots$
4	$-\frac{1}{16}$	$\vdots$	

The eigenvalues λ correspond to the energy levels. For example, $\lambda = -1$ is the lowest energy level. That they're negative doesn't matter since they're relative. The reason that energy levels are quantized is because the eigenfunctions can only take certain values. The λ gives the emission spectra of hydrogen.

Remember, these are only the spherically symmetric solutions, which correspond to the s-shells of the hydrogen atom. There are also other solutions, which correspond to the p, d, and f subshells.

Other Solutions

Here we look for solutions to the eigenvalue equation

$$-\Delta X - \frac{2}{r}X = \lambda X$$

that depend on the angles in addition to r. We start by separating the variables as

$$X = R(r)\cdot Y(\theta,\phi).$$

Taking the relevant derivatives including the Laplacian for spherical coordinates, simplifying and rearranging gives us

$$\begin{aligned} 0 &= \lambda r^2 + 2r + \frac{r^2R_{rr}+2rR_r}{R} \\ &\quad + \frac{1}{Y}\left[\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}\left(\sin\phi Y_\phi\right)_\phi\right]. \end{aligned}$$

Notice that the first three terms depend only on r, and the remaining terms depend only on the angles. Since their sum equals zero, they are both constants, which we choose to be γ, so

$$\begin{aligned} \gamma &= \lambda r^2 + 2r + \frac{r^2R_{rr}+2rR_r}{R} \\ -\gamma &= \frac{1}{Y}\left[\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}\left(\sin\phi Y_\phi\right)_\phi\right]. \end{aligned}$$

The angular equation

$$\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}\left(\sin\phi Y_\phi\right)_\phi + \gamma Y = 0,$$

is the standard spherical harmonics problem which we have solved before. The solutions are

$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\phi)e^{im\theta},$$

where $l = 0, 1, 2, \ldots$ is the **orbital quantum number** and $m = -l, \ldots, l$ is the **magnetic quantum number**, and $\gamma = l(l+1)$.

The R-equation is

$$R_{rr} + \frac{2}{r}R_r + \left[\lambda + \frac{2}{r} - \frac{l(l+1)}{r^2}\right]R = 0.$$

The spherically-symmetric case occurs when $l = 0$.

Making the substitution

$$R = w(r)e^{-\beta r},$$

where $\lambda = -\beta^2$ with $\beta > 0$, gives us

$$w_{rr} + 2\left(\frac{1}{r} - \beta\right)w_r + \left(\frac{2(1-\beta)}{r} - \frac{l(l+1)}{r}\right)w = 0.$$

Next we look for a series solution of the form

$$w(r) = \sum a_k r^k.$$

Taking the relevant derivatives, plugging it in, simplifying, and shifting the index on one of the sums gives us

$$\begin{aligned} 0 &= -l(l+1)a_0 r^{-2} \\ &\quad + \sum_{k=1}^{\infty} \left[k(k+1) - l(l+1)\right] a_k r^{k-2} \\ &\quad + \sum_{k=1}^{\infty} 2(1-\beta k) a_{k-1} r^{k-2}. \end{aligned}$$

Since the right side is zero, all of the coefficients must be zero. Since $l \neq 0$, the first term implies that $a_0 = 0$. The rest of it gives us the recursion equation

$$2(1-\beta k)a_{k-1} = \left[k(k+1) - l(l+1)\right] a_k.$$

When $k = 1$, we get

$$2(1-\beta)a_0 = \left[2 - l(l+1)\right] a_1.$$

Since $a_0 = 0$, the left side is zero, and so the right side is also zero. This implies that either $a_1 = 0$ in which case every term is zero because of the recursion relationship or $l = 1$. If we write out a few more of the above equations for $k = 2, 3, 4, \ldots$, it is easy to see that $a_0 = 0$ implies that all the coefficients up to and including a_{l-1} are zero. Then a_l is the first nonzero term. So if $l = 5$, then a_5 is the first nonzero term. In order for the series to terminate, which we need in order for the solution to decay at infinity, we also need $\beta = \frac{1}{n}$ where $n = l+1, l+2, \ldots$

What we end up getting is

$$w(r) = L_n^l(r),$$

a polynomial related to the **associated Laguerre polynomials**.

So our eigenvalue solution is

$$\begin{aligned} X_{nlm} &= e^{-\frac{r}{n}} L_n^l(r) Y_l^m(\theta, \phi) \\ &= e^{-\frac{r}{n}} L_n^l(r) P_l^{|m|}(\cos\phi) e^{im\theta}. \end{aligned}$$

The eigenvalues are

$$\lambda_{nlm} = -\frac{1}{n^2},$$

The number $n = 1, 2, 3, \ldots$ is the **principal quantum number**, $l = 0, 1, \ldots, n-1$ is the **orbital quantum number**, and $m = -l, \ldots, l$ is the **magnetic quantum number**.

Below is a table showing the solutions for different values of the quantum numbers n, l, and m. With a little deduction, the shapes made by these functions can be determined. It turns out that when $l = 0$, you get the spherically-symmetric s-shells. When $l = 1$, you get the p-shells. When $l = 2$, you get the d-shells.

n	l	m	X_{nlm}
1	0	0	e^{-r}
2	0	0	$e^{-\frac{r}{2}}\left(1 - \frac{1}{2}r\right)$
2	1	0	$e^{-\frac{r}{2}} r\cos\phi$
2	1	± 1	$e^{-\frac{r}{2}} r\sin\phi \left\{\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right\}$
3	0	0	$e^{-\frac{r}{3}}\left(1 - \frac{2}{3}r + \frac{2}{27}r^2\right)$
3	1	0	$e^{-\frac{r}{3}}\left(r - \frac{1}{6}r^2\right)\cos\phi$
3	1	± 1	$e^{-\frac{r}{3}}\left(r - \frac{1}{6}r^2\right)\sin\phi \left\{\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right\}$
3	2	0	$e^{-\frac{r}{3}} r^2(3\cos^2\phi - 1)$
3	2	± 1	$e^{-\frac{r}{3}} r^2 \cos\phi\sin\phi \left\{\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right\}$
3	2	± 2	$e^{-\frac{r}{3}} r^2 \sin^2\phi \left\{\begin{smallmatrix}\cos 2\theta \\ \sin 2\theta\end{smallmatrix}\right\}$

1.10 The Method of Characteristics

1.10.1 Theory

Given a first order PDE, for example, $u_t + uu_x = 0$ or $u_x u_y = u$, we can write the PDE in the form

$$F(u_{x_1}, u_{x_2}, \ldots, u_{x_n}, u, x_1, x_2, \ldots, x_n) = 0.$$

In other words, if we have a PDE of the function u and its partial derivatives u_{x_i} where u is a function of n variables x_i, then we can write the PDE as a function F of the form given above. We include the variables x_i in F, since the PDE can include the variables themselves. Note, *linear* in this case means the partial derivatives are not raised to powers.

To shorten this notation, we let $p_i = u_{x_i}$ so that our PDE becomes

$$F(p_1, p_2, \ldots, p_n, u, x_1, x_2, \ldots, x_n) = 0.$$

We want to find a curve $\vec{\boldsymbol{x}}(s) = (x_1(s), x_2(s), \ldots, x_n(s))$ such that the PDE becomes an ODE along this curve. Such a curve is called a **characteristic curve**. In general, the characteristic curve will be in n-dimensions, and we parametrize it with the parameter s. Supposing that the PDE has a solution, then we have $u(\vec{\boldsymbol{x}}(s)), p_1(\vec{\boldsymbol{x}}(s)), p_2(\vec{\boldsymbol{x}}(s)), \ldots, p_n(\vec{\boldsymbol{x}}(s))$ all defined along our characteristic curve. Also, we know that

$$F(p_1(\vec{\boldsymbol{x}}(s)), \ldots, p_n(\vec{\boldsymbol{x}}(s)), u(\vec{\boldsymbol{x}}(s)), x_1(s), \ldots, x_n(s)) = 0.$$

Differentiating $p_i(\vec{\boldsymbol{x}}(s))$ gives us

$$\begin{aligned}
\dot{p}_i &= \frac{d}{ds}(u_{x_i}(\vec{\boldsymbol{x}}(s))) \\
&= \frac{d}{ds}(u_{x_i}(x_1(s), x_2(s), \ldots, x_n(s))) \\
&= u_{x_i x_1}(\vec{\boldsymbol{x}}(s))\dot{x}_i + \cdots + u_{x_i x_n}(\vec{\boldsymbol{x}}(s))\dot{x}_n \\
&= \sum_{j=1}^{n} u_{x_i x_j}\, \dot{x}_j.
\end{aligned}$$

We had to use the multi-dimensional chain rule. Also notice that we are using the dot notation to mean the derivative with respect to s.

If the PDE F is defined everywhere, we can differentiate it with respect to each variable x_i. Remember that each of the partial derivatives $u_{x_1}, u_{x_2}, \ldots, u_{x_n}$ may be functions of x_i, so we have to use the multidimensional chain rule. Differentiating F with respect to x_i gives us

$$\frac{\partial F}{\partial p_1} u_{x_1 x_i} + \cdots + \frac{\partial F}{\partial p_n} u_{x_n x_i} + \frac{\partial F}{\partial u} u_{x_i} + \frac{\partial F}{\partial x_i} = 0.$$

That is,

$$\sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_j} u_{x_j x_i} + \frac{\partial F}{\partial u} u_{x_i} + \frac{\partial F}{\partial x_i} \right) = 0.$$

Recall that

$$\dot{p}_i = \sum_{j=1}^{n} u_{x_i x_j}\, \dot{x}_j.$$

To make these sums similar, we choose the characteristic curve so that

$$\dot{x}_j = \frac{\partial F}{\partial p_j}.$$

This is now an ODE. We can now combine the two series to get

$$\dot{p}_i = -\frac{\partial F}{\partial u} u_{x_i} - \frac{\partial F}{\partial x_i} = -\frac{\partial F}{\partial u} p_i - \frac{\partial F}{\partial x_i},$$

and

$$\dot{u} = \sum_{j=1}^{n} u_{x_j} \dot{x}_j = p \nabla_p F.$$

We now have a system of $2n + 1$ ODEs, where n is the number of dimensions. To get specific solutions, we need one initial condition for each characteristic curve, so in the 2D case, we need a curve, and in the 3D case, we need a surface on which the initial conditions are specified.

1.10.2 Simple Example

Solve the PDE

$$xu_y - yu_x = u,$$

with initial conditions $u(x, 0) = x^2$ for $x > 0$ using the method of characteristics.

Step 1: Find F

The first thing we have to do is find F. Note that $u_x = p_1$ and $u_y = p_2$, but to keep things simple, we let $u_x = p$ and $u_y = q$.

To find F we replace the partial derivatives by the p_i and move everything to one side since F is equal to zero. We find that F is

$$F = xq - yp - u = 0.$$

Step 2: Set up ODEs

In the second step, we set up the system of ODEs. Our F has 5 variables, so we get a system of 5 ODEs.

$$\begin{aligned}
\dot{x} &= \frac{\partial F}{\partial p} = -y \\
\dot{y} &= \frac{\partial F}{\partial q} = x \\
\dot{u} &= p\nabla_p F = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} = -py + qx \\
\dot{p} &= -\frac{\partial F}{\partial u}p - \frac{\partial F}{\partial x} = p - q \\
\dot{q} &= -\frac{\partial F}{\partial u}q - \frac{\partial F}{\partial y} = q + p.
\end{aligned}$$

Notice that from F, we can see that $\dot{u} = -py + qx = u$. Since $\dot{x}, \dot{y}$, and $\dot{u}$ do not depend on p or q anymore, we don't actually need the last two equations above to solve for u. We only need to solve the following system of ODEs

$$\begin{aligned}
\dot{x} &= -y \\
\dot{y} &= x \\
\dot{u} &= u.
\end{aligned}$$

Step 3: Initial Conditions

To handle the initial conditions, we have to parametrize them as well. We parametrize it with a new variable a. Since we are given the initial conditions in the form $u(x, 0)$ for $x > 0$, which in the xy-plane is the ray extending along the positive x axis, we parametrize such that a tells us where we are along this ray. For each point a along the initial conditions curve, we have a characteristic curve. In other words, (r, s) gives us a unique characteristic curve given by r and a position s along that curve.

By convenience, when $s = 0$, we are *on* the initial conditions curve. Then

$$\begin{aligned}
x(0) &= a \\
y(0) &= 0 \\
u(0) &= a^2.
\end{aligned}$$

The value of $u(0)$ is obtained from the initial conditions $u(x, 0) = x^2$.

Step 4: Solve the System of ODEs

Now we solve the system of ODEs

$$\begin{aligned}
\dot{x} &= -y \\
\dot{y} &= x \\
\dot{u} &= u.
\end{aligned}$$

The u equation is easy to solve. It's solution is $u(s) = Ce^s$. From the initial condition $u(0) = a^2$, we get that $C = a^2$, so

$$u(s) = a^2 e^s.$$

If we differentiate the x equation again, we get $\ddot{x} = -\dot{y}$, and we know that $\dot{y} = x$, so we have that

$$\ddot{x} = -x.$$

This ODE has the solution $x(s) = C_1 \cos x + C_2 \sin x$. From the initial conditions $x(0) = a$, we get that $C_1 = a$ and $C_2 = 0$, so

$$x(s) = a \cos s.$$

Since $y = -\dot{x}$, we can just differentiate $x(s)$ to get

$$y(s) = a \sin s.$$

Step 5: Invert the Solution

Essentially, we are done, but we have the wrong variables. We need to invert x, y, and u so that we get the solution u as a function of x and y rather than a and s.

In this case, we recognize $x(s)$ and $y(s)$ as the polar transformations. So

$$\begin{aligned}
a &= \sqrt{x^2 + y^2} \\
s &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}$$

So our solution can be written as

$$u(x, y) = (x^2 + y^2)e^{\tan^{-1}(y/x)}.$$

In principle, the method of characteristics works for any first-order PDE. However, the resulting system of ODEs may be impossible to solve analytically, or when inverting the x and y equations to get r and s, you might run into algebra problems.

Step 6: Check the Solution

The final step is to check the solution against the original PDE and initial conditions. Taking the partial derivatives, we get

$$\begin{aligned}
u_x &= (2x - y)e^{\tan^{-1}(y/x)} \\
u_y &= (x + 2y)e^{\tan^{-1}(y/x)}.
\end{aligned}$$

Plugging them into the PDE and simplifying verifies that our solution works. Plugging in $y = 0$, gives us

$$u(x, 0) = (x^2 + (0)^2)e^{\tan^{-1}(0/x)} = x^2.$$

which verifies the initial conditions. So our solution is good.

1.10.3 Example

Solve the PDE

$$u_x^2 + yu_y = u,$$

with initial conditions $u(x,1) = \frac{1}{4}x^2 + 1$ using the method of characteristics.

Step 1: Find F

The first thing we have to do is find F. Let $u_x = p$ and $u_y = q$, then

$$F = p^2 + yq - u = 0.$$

Step 2: Set up ODEs

In the second step, we set up the system of ODEs. Our F has 5 variables, so we get a system of 5 ODEs.

$$\begin{aligned}
\dot{x} &= \frac{\partial F}{\partial p} = 2p \\
\dot{y} &= \frac{\partial F}{\partial q} = y \\
\dot{u} &= p\nabla_p F = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} = 2p^2 + yq \\
\dot{p} &= -\frac{\partial F}{\partial u}p - \frac{\partial F}{\partial x} = p \\
\dot{q} &= -\frac{\partial F}{\partial u}q - \frac{\partial F}{\partial y} = 0.
\end{aligned}$$

Because of the partial derivative that is squared in the original PDE, this PDE is not semilinear, and $\dot{u} \neq u$. This means we have to use the p and q equations too.

Step 3: Initial Conditions

The initial conditions are $u(x,1) = \frac{1}{4}x^2 + 1$. We parametrize them by letting $x = a$ when $s = 0$. So when $s = 0$,

$$\begin{aligned}
x(0) &= a \\
y(0) &= 1 \\
u(0) &= \frac{1}{4}a^2 + 1 \\
p(0) &= \frac{1}{2}a \\
q(0) &= 1.
\end{aligned}$$

We have to work a litte more to find the last two. Recall that $p = u_x$, so by differentiating the initial conditions, we get

$$p = u_x = \frac{\partial}{\partial x}\left(\frac{1}{4}x^2 + 1\right) = \frac{1}{2}x.$$

Plugging in $x = a$ gives us $p(0) = \frac{1}{2}a$. To find $q(0)$ we can use the PDE since we know all the other values in the PDE.

$$0 = \left(\frac{1}{2}a\right)^2 + q - \left(\frac{1}{4}a^2 + 1\right) = q - 1,$$

so $q = 1$.

Step 4: Solve the System of ODEs

Now we solve the system of ODEs

$$\begin{aligned}
\dot{x} &= 2p \\
\dot{y} &= y \\
\dot{u} &= 2p^2 + yq \\
\dot{p} &= p \\
\dot{q} &= 0.
\end{aligned}$$

We start with the easiest ones and substitute them into the others as we go along. The solution to the q ODE is $q(s) = C$. From the initial conditions, we have that $C = 1$, so $q(s) = 1$. Similarly, the solution to the y ODE is $y(s) = e^s$, and the solution to the p ODE is $p(s) = \frac{1}{2}ae^s$. Substituting these results into the remaining two problems gives us

$$\begin{aligned}
\dot{x} &= ae^s \\
\dot{u} &= \frac{1}{2}a^2e^{2s} + e^s.
\end{aligned}$$

To solve these, we now just have to integrate both with respect to s and then use the initial conditions to find the constants. All together, our solutions are

$$\begin{aligned}
x &= ae^s \\
y &= e^s \\
u &= \frac{1}{4}a^2e^{2s} + e^s \\
p &= \frac{1}{2}ae^s \\
q &= 1.
\end{aligned}$$

Step 5: Invert the Solution

We need the solution in terms of x and y, so if we manipulate the x and y solutions above, we get $e^s = y$, which means we can rewrite $x = ae^s$ as $x = ay$, or $a = \frac{x}{y}$. Replacing a and e^s in u, gives us

$$u(x,y) = \frac{1}{4}\left(\frac{x}{y}\right)^2 y^2 + y = \frac{1}{4}x^2 + y.$$

1.11 Summary

1.11.1 Fourier Series

If $f(x)$ is piecewise smooth, then its Fourier series converges to the periodic extension of $f(x)$ at every point x at which $f(x)$ is continuous. At discontinuities, the Fourier series converges to the average of the values immediately to the left and right of the discontinuities.

A Fourier series can always be integrated. It can also be differentiated provided the graph is continuous.

To remember the different formulas for the different Fourier series, at least memorize the formula for the full Fourier series. The sine series can easily be derived from this by treating $f(x)$ as an odd function and realizing that the A_n terms will all be zero and the integral for the B_n is multiplied by two but integrated over $[0, L]$ instead of $[-L, L]$. Similarly the cosine series can easily be derived by treating $f(x)$ as an even function and realizing that the B_n terms will all be zero, and the integrals for the A_n coefficients is multiplied by 2 but integrated only over $[0, L]$.

To remember the factors in front of the integral formulas for the Fourier series, remember that A_0 is the average value of the function over the interval, and A_n and B_n have a factor that is two times the factor in front of A_0.

The following trigonometric identities are often useful when performing the integrations to find the Fourier coefficients.

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\cos^2 x &= \frac{1}{2} + \frac{1}{2}\cos 2x \\
\sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos 2x \\
\sin 2x &= 2\sin x\cos x.
\end{aligned}$$

By substituting $-B$ for B in the first two, you can easily find the formulas for $\sin(A-B)$ and $\cos(A-B)$.

The following simplifications should also be remembered

$$\begin{aligned}
\cos(n\pi) &= (-1)^n \\
\sin(n\pi) &= 0 \\
e^{in\pi} &= (-1)^n \\
e^{-in\pi} &= (-1)^n.
\end{aligned}$$

Full Series

The full Fourier series of a function $f(x)$ defined on $[-L, L]$ is

$$\begin{aligned}
f(x) &= A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right] \\
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.
\end{aligned}$$

To graph the full Fourier series of $f(x)$ on $[-L, L]$,

1. Graph $f(x)$ on $[-L, L]$
2. Graph the periodic extension of the above. That is, repeat the above segment to the left and right.
3. Plot the defined points at any discontinuities.

Because of the properties of the definite integrals of odd and even functions, if $f(x)$ is odd, the full Fourier series reduces to the sine series, and if $f(x)$ is even it reduces to the cosine series.

Fourier Sine Series

The Fourier sine series of $f(x)$ defined on $[0, L]$ is

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\
B_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.
\end{aligned}$$

To graph the Fourier sine series of $f(x)$ on $[0, L]$,

1. Graph $f(x)$ on $[0, L]$
2. Graph the odd extension of the above on $[-L, 0]$
3. Graph the periodic extension of the above two to the left and right
4. Plot the points at the discontinuities

Fourier Cosine Series

The Fourier cosine series of $f(x)$ defined on $[0, L]$ is

$$\begin{aligned}
f(x) &= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) \\
A_0 &= \frac{1}{L}\int_{0}^{L} f(x)\,dx \\
A_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.
\end{aligned}$$

To graph the Fourier cosine series of $f(x)$ on $[0, L]$,

1. Graph $f(x)$ on $[0, L]$
2. Graph the even extension of the above on $[-L, 0]$
3. Graph the periodic extension of the above two to the left and right
4. Plot the points at the discontinuities

Notice that A_0 is just the average value of $f(x)$.

The cosine series of a constant function is just the constant.

Complex Fourier Series

The complex Fourier expansion for a function $f(x)$ defined on $[-L, L]$ is given by

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}} \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}}\, dx. \end{aligned}$$

Notice that the signs in the exponent are not the same in the sum and the integral, and that the lower limit on the series is $n = -\infty$.

1.11.2 The Heat Equation

The heat equation is

$$\frac{\partial u}{\partial t} = k\nabla^2 u + q.$$

Dirichlet BCs: When the temperature at the boundary is specified

$$\begin{aligned} u(0,t) &= \alpha(t) \\ u(L,t) &= \beta(t). \end{aligned}$$

Neumann BCs: When the flux at the boundary is specified

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \alpha(t) \\ \frac{\partial u}{\partial x}(L,t) &= \beta(t). \end{aligned}$$

Periodic BCs: When the temperature and flux at one boundary is the same as that at the opposite boundary

$$\begin{aligned} u(-L,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t). \end{aligned}$$

The homogeneous heat equation with homogeneous boundary conditions can be solved by separation of variables. Otherwise, use eigenfunction expansion.

Equilibrium Solutions

Equilibrium or steady-state solutions are solutions that do not depend on t. That is, if the system has one or more equilibriums, then there will be one or more solutions which do not depend on t. To find them, set $\frac{\partial u}{\partial t} = 0$, and solve the simplified heat equation

$$\nabla^2 u = -\frac{q}{k}.$$

For the one-dimensional case, this simplifies to the ordinary separable differential equation

$$\frac{d^2u}{dx^2} = -\frac{q}{k}.$$

The general process for finding equilibrium solutions is

1. Let $u_t = 0$ in the heat equation
2. Solve the resulting separable ODE
3. Use the boundary conditions to find the resulting pair of constants. If you get two different values for the same constant, then no equilibrium solution exists.
4. If the boundary conditions do not specify one of the constants, which may occur with Neumann or periodic boundary conditions, then the remaining constant is a function of the average value of the initial conditions $u(x, 0) = f(x)$. To determine the function, use the fact that an equilibrium solution $u(x)$ (which contains the unknown constant) implies that for large t, $u(x, t) \approx u(x)$, and *if the total amount of heat in the wire is constant*, then

   $$\int_0^L u(x,t)\, dx = \int_0^L f(x)\, dx = \int_0^L u(x)\, dx.$$

   Note, for periodic BCs, the lower limits on the integral are $-L$ instead of 0.

1.11.3 Separation of Variables

Separation of variables can be used to solve some linear homogeneous PDEs. The steps to follow are

1. Separate the the variables to obtain two ordinary differential equations. This means writing the dependent variable in terms of the product of functions of the independent variables. For the one-dimensional heat equation, the dependent variable is u and the independent variables are x and t, so you would write

   $$u(x,t) = X(x) \cdot T(t).$$

   Plug this back into the PDE and separate the variables so you end up with a function only of x on one side and a function only of t on the other side. Then both sides must be equal to a constant, so the PDE can be split into to ordinary differential equations.

2. Find the nontrivial solutions to the eigenvalue problem using the boundary conditions. This is the ordinary differential equation from the previous step that is in the homogeneous direction. For the one-dimensional heat equation, this is typically the ODE containing x. Solving this ODE amounts to finding the eigenvalues and eigenfunctions of the solution to the heat equation.
3. Solve the other ODE using the eigenvalues found in the previous step.
4. The general solution is the linear combination of the product of the solutions to the two ODEs solved in the previous two steps. Note that this is an infinite sum.
5. Use the initial conditions $u(x,0) = f(x)$ to compute the coefficients in the general solution.

Below is a table with the information to solve the eigenvalue problem

$$X''(x) + \lambda X(x) = 0,$$

for the homogeneous heat equation.

	Dirichlet BCs	Neumann BCs	Periodic BCs
Initial Conditions	$u(x,0) = f(x)$	$u(x,0) = f(x)$	$u(x,0) = f(x)$
Boundary Conditions	$X(0) = 0$, $X(L) = 0$	$X'(0) = 0$, $X'(L) = 0$	$X(-L) = X(L)$ $X'(-L) = X'(L)$
Eigenvalues, λ_n	$\left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 1, 2, 2, 3, 3, \ldots$
Eigenfunctions, $X_n(x)$	$\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$	$\cos\left(\frac{n\pi x}{L}\right)$, $X_0(x) = 1$, $n = 0, 1, 2, 3, \ldots$	$\cos\left(\frac{n\pi x}{L}\right)$, $\sin\left(\frac{n\pi x}{L}\right)$, $X_0(x) = 1$
Fourier Series of Initial Conditions	Sine Series	Cosine Series	Full series

1.11.4 Laplace's Equation

Laplace's Equation on a Rectangle

In Cartesian coordinates, Laplace's equation is

$$u_{xx} + u_{yy} = 0.$$

We are given the temperature (Dirichlet BCs) or the flux (Neumann BCs) on the four sides of the rectangle. We can solve this problem using separation of variables by solving for one side of the rectangle at a time. That is, we set all but one of the sides equal to zero then treat the nonzero side as the "initial conditions". The complete solution is the sum of the four solutions which each treat a different side as nonzero.

In the first step, when separating the variables, set the sign on the constant λ such that you end up with an eigenvalue equation of the form $X'' + \lambda X = 0$ or $Y'' + \lambda Y = 0$, an ODE that we have solved before and know the solution to. To do this, you must note the homogeneous direction. If the BCs at the top and bottom of the rectangle are zero, then y is the homogeneous direction and you want the eigenvalue equation to be of the form $Y'' + \lambda Y = 0$. If the BCs at the left and right sides of the rectangle are zero, then x is the homogeneous direction and you want the eigenvalue equation to be of the form $X'' + \lambda X = 0$.

Laplace's Equation on a Disk

For Laplace's equations on a disk, the Laplace equation, the boundary conditions, and the initial conditions must all be converted to polar coordinates before doing anything else. Laplace's equation in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Separating the variables gives the ODEs

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ r^2R'' + rR' - \lambda R &= 0. \end{aligned}$$

If it's on a full disk, the boundary conditions are periodic and on $[-\pi, \pi]$. If it's on a half or quarter disk, then the straight boundaries will be specified and on $[0, \pi]$ or $[0, \pi/2]$.

Then θ is taken to be the homogeneous direction, and is solved as in previous examples since the eigenvalue equation is the same. The other ODE is

$$r^2R'' + rR' - n^2R = 0.$$

Since $R(0)$ must be finite, the solution is

$$R_n(r) = C_1r^n, \qquad R_0(r) = C_1.$$

In the end, we use the remaining boundary condition, the temperature along the outer edge of the disk, to compute the Fourier coefficients.

If we can, we convert the final answer back to cartesian coordinates using the cartesian/polar transformation equations.

1.11.5 Eigenfunction Expansion

Eigenfunction expansion can be used to solve nonhomogeneous PDEs.

Step 1

The first step is to write the solution, all functions of x in the original PDE, and the initial conditions as Fourier series. Where possible, simplify these in terms of the Fourier series of the solution or in terms of the boundary conditions. The kind of Fourier series to use depends on the boundary conditions. For example, if the boundary conditions are Dirichlet-like, then use a sine series for all the Fourier series.

Remember that when computing the Fourier series with respect to one variable of a function of two variables, the coefficients will be functions of the other variable. For example, the Fourier series of $u(x,t)$ with respect to x will have coefficients which are functions of t.

Step 2

Substitute the Fourier series for each term in the PDE back into the PDE. Rearrange the result so that all terms are on the left side and the right side is zero. Rearrange the left side, factoring as necessary to get it into the form of a Fourier series. The left side should now be a single Fourier series, and the right side is the corresponding Fourier series of 0. Now, equate the corresponding coefficients on either side, the right side being all zeros. This results in the original PDE being reduced to nonhomogeneous ODEs with respect to t.

Step 3

Solve the ODE(s) found in the previous step.

If the original PDE is the heat equation, the resulting ODE will be of the form

$$\frac{d}{dt}\hat{u}_n + M\hat{u}_n(t) = \hat{r}_n(t),$$

where $\hat{r}_n(t)$ is a known function of t. This ODE can be solved using the method of an integrating factor. The integrating factor is

$$\text{I.F.} = e^{\int M\,dt}.$$

Next, multiply both sides of the ODE by the integrating factor, so that the left side is now the result of the product rule. Finally, do a definite integral from 0 to t of both sides and solve for the coefficients of the solution $\hat{u}_n(t)$.

1.11.6 Some ODEs and their Solutions

Remember that the ODEs coming from the equilibrium solutions are all separable, and thus, easily solved just by integrating.

Note, the constants appearing in one equation are generally not the same as the constants appearing in another equation.

Example 1

$$X'' + \lambda X = 0$$

Solutions are of the form:

$$\begin{aligned} X(x) &= C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x} \\ X(x) &= C_1\cos\beta x + C_2\sin\beta x, \quad \text{If } \lambda = \beta^2 > 0 \\ X(x) &= C_1 + C_2 x, \quad \text{If } \lambda = 0 \\ X(x) &= C_1\cosh\gamma x + C_2\sinh\gamma x, \quad \text{If } \lambda = -\gamma^2 < 0 \end{aligned}$$

Example 2

$$T' + k\lambda T = 0,$$

has the general solution

$$T(t) = Ce^{-k\lambda t}$$

Example 3

$$X'' - \lambda X = 0.$$

Solutions are of the form:

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x} \\ X(x) &= C_1\cosh\beta x + C_2\sinh\beta x, \quad \text{If } \lambda = \beta^2 > 0 \\ X(x) &= C_1\cosh(\beta[x-L]) \\ &\quad + C_2\sinh(\beta[x-L]), \quad \text{If } \lambda = \beta^2 > 0 \end{aligned}$$

Example 4

$$r^2R'' + rR' - n^2R = 0.$$

The general solution is

$$\begin{aligned} R_n(r) &= C_1 r^n + C_2 r^{-n}, \quad \text{when } n \neq 0 \\ R_0(r) &= C_1 + C_2\ln r, \quad \text{when } n = 0. \end{aligned}$$

1.11.7 Wave Equation

Solve the wave equation like you would the heat equation. Use separation of variables for the homogeneous case. For the nonhomogeneous wave equation, use eigenfunction expansion.

1.11.8 Sturm-Liouville Eigenvalue Problems

A **Sturm-Liouville** eigenvalue problem has the form

$$(pu')' + qu + \lambda\sigma u = 0.$$

The eigenfunctions are $u(x)$, and the eigenvalues λ are the unknowns. The variables p, q, and σ are known functions of x on $a \le x \le b$. All eigenvalue problems that come up, in practice, are of this form.

In general, we will assume that p, q, and σ are nice functions. This means $p(x), \sigma(x) > 0$ for $a \le x \le b$ and $p(x)$, $q(x)$, and $\sigma(x)$ are continuous for $a \le x \le b$. If these conditions are met, the problem is *regular*. Otherwise, the problem is *singular*.

For Sturm-Liouville eigenvalue problems, the eigenvalues and eigenfunctions are real. The eigenfunctions are *complete*, so any reasonable function $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x).$$

The eigenfunctions are orthogonal with respect to $\sigma(x)$. That is

$$\int_a^b u_n(x)u_m(x)\sigma(x)\,dx = 0, \quad \text{if } \lambda_m \neq \lambda_n,$$

so

$$a_n = \frac{\int_a^b f(x)u_n(x)\sigma(x)\,dx}{\int_a^b u_n^2(x)\sigma(x)\,dx}.$$

If we start with the general eigenvalue problem

$$(pu')' + qu + \lambda\sigma u = 0.$$

then multiply both sides by u and integrate from a to b, we get

$$\begin{aligned} \int_a^b u\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + q\phi + \lambda\sigma u\right]dx &= 0 \\ \int_a^b \frac{d}{dx}\left(p\frac{du}{dx}\right)u\,dx + \int_a^b qu^2\,dx + \lambda\int_a^b u^2\sigma\,dx &= 0. \end{aligned}$$

Finally, we solve for λ to get the Rayleigh quotient

$$\lambda = \frac{-\int_a^b \frac{d}{dx}\left(p\frac{du}{dx}\right)u\,dx - \int_a^b qu^2\,dx}{\int_a^b u^2\sigma\,dx}.$$

Integrating by parts gives us

$$\lambda = \frac{-pu\frac{d\phi}{dx}\Big|_{x=a}^{b} + \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - qu^2\right]dx}{\int_a^b u^2\sigma\,dx}.$$

Notice that if $-pu\frac{du}{dx}\Big|_{x=a}^{b} \ge 0$ and $q(x) \le 0$, then every term in the equation above is positive and so $\lambda > 1$.

The Rayleigh quotient of an eigenfunction gives the associated eigenvalue. That is,

$$RQ[u_n] = \lambda_n.$$

We can approximate the first eigenvalue by choosing a nice function $u(x)$ that satisfies the boundary conditions and taking its Rayleigh quotient. That is,

$$RQ[u] = \frac{-puu'\Big|_a^b + \int_a^b \left[p\,(u')^2 - qu^2\right]dx}{\int_a^b u^2\sigma\,dx} \ge \lambda_1.$$

If $f(x)$ on $[a, b]$, is defined by the Fourier series

$$f(x) = \sum_{n=1}^{\infty} \hat{f}_n\phi_n(x),$$

where $\phi_n(x)$ are eigenfunctions, then by the orthogonality of eigenfunctions, it follows that

$$\hat{f}_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx},$$

then **Bessel's inequality** tells us that

$$\sum_{n=1}^{\infty} |\hat{f}_n|^2 \int_a^b \phi_n^2(x)\sigma(x)\,dx \le \int_a^b |f(x)|^2\sigma(x)\,dx.$$

In the case of equality, the above is called **Parseval's equation**.

Given a Fourier series, to "apply" Parseval's equation means to take the absolute value of both sides, square both sides, then integrate both sides over the interval of definition. For example, given the Fourier sine series for a function defined on $[0, L]$

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

applying Parseval's equation means to simplify the following

$$\int_0^L |f(x)|^2\,dx = \sum_{n=0}^{\infty} |b_n|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx.$$

For Sturm-Liouville problems, we always get equality in Bessel's inequality.

1.11.9 Higher-dimensional PDEs

A higher-dimensional PDE is a partial differential equation in which the solution is defined over more than one dimension. For example, the solution might be defined over a rectangle, a disk, or a sphere.

In polar coordinates, the Laplacian is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

To remember the form of this, remember that there are two r's in every term (if you think of r^2 as rr), and it goes from second derivative in the first term to the first derivative in the second term to the zeroth derivative with respect to r in the third term.

To solve a higher dimensional equation using separation of variables,

1. Separate the space and time variables
2. Separate the eigenvalue equation. The eigenvalue equation is typically the spatial equation, which for higher dimensions involves more than just x. You want to separate this PDE into an ODE in each variable. For example, if it's over a disk, you would separate it using polar coordinates, and if it's a sphere, you would separate it using spherical coordinates. You then solve the resulting ODEs, getting the eigenvalues from the ODE in the homogeneous direction.
3. Solve the time equation with the eigenvalues found in the previous step.
4. Multiply all the solutions together and sum over them to get the general solution.
5. Solve for the coefficients in the series solution using the given initial conditions and the orthogonality of the eigenfunctions.

Bessel's equation of order n is the ODE

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right)R = 0.$$

Its solution is a linear combination of the **Bessel functions**

$$R(r) = C_1 J_n(\sqrt{\lambda}r) + C_2 Y_n(\sqrt{\lambda}r).$$

Each of these terms gives a decaying oscillation, but the key difference is that the first term is finite at $R(0)$ and the second term is infinite. So if the solution needs to be finite at $R(0)$, then the second coefficient must be zero so that the second term is not there.

If $R(0) < \infty$ and $R(a) = 0$, then the solutions are

$$R(r) = J_n(\sqrt{\lambda}r),$$

where

$$\lambda_{nm} = \left(\frac{z_{nm}}{a}\right)^2,$$

and z_{nm} is the mth zero of $J_n(\sqrt{\lambda}r)$.

1.11.10 The Method of Characteristics

To use the method of characteristics, we follow the following procedure:

1. Find F by moving everything in the PDE to one side, setting it equal to zero, and substituting p_i for the partial derivatives. For example, $p_1 = u_x$ and $p_2 = u_y$.
2. Set up the system of ODEs using the formulas:

$$\begin{aligned} \dot{x}_i &= \frac{\partial F}{\partial p_i} \\ \dot{u} &= \sum p_i \dot{x}_i \\ \dot{p}_i &= -\frac{\partial F}{\partial u}p_i - \frac{\partial F}{\partial x_i}. \end{aligned}$$

   If $\dot{u}$ simplifies to u, and $\dot{x}_i$ does not involve p_i, then you can ignore the $\dot{p}_i$ equation(s).
3. Parametrize the initial conditions with a new variable(s). For $p_1(0)$, differentiate the initial conditions. For the other $p_i(0)$ you have to develop a system of equations including the original PDE.
4. Solve the system of ODEs given the parametrized initial conditions. Remember these will be functions of your parameter(s) , usually just s.
5. Invert the x_i solutions in order to get the solution to the PDE in terms of the variables you want.
6. Check the solution by taking the appropriate derivatives and plugging it into the PDE and by plugging in the initial conditions.

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